Proceedings of the International Workshop
Future Directions in Difference Equations.
June 13-17, 2011, Vigo, Spain.
PAGES 149–156

Set-oriented numerical analysis of difference equations with parameters

Paweł Pilarczyk

University of Minho, Braga, Portugal

http://www.pawelpilarczyk.com/

Abstract

The purpose of this paper is to provide a possibly simple introduction to the set-oriented algorithmic method for determining the global dynamics of a system of first-order ordinary difference equations with parameters, introduced in the paper "A database schema for the analysis of global dynamics of multiparameter systems" (SIADS 2009, Vol. 8, No. 3, pp. 757–789). An emphasis is made on the details of input data and on the assumptions required by the method, as well as on the interpretation of the results of the computation. The method is illustrated with an application to a simple overcompensatory two-stage population model related to modeling baleen whale populations.

1 Introduction

Given a system of first-order ordinary difference equations with a few parameters, a very natural question to ask is what the long-term qualitative behavior of solutions to the system is, and how this behavior changes when the parameters are varying. The first and obvious attempt to answer this question is to solve the equations for equilibria and to analytically investigate their stability by means of linearization. Indeed, the information obtained in this way provides a first insight into the global dynamics represented by the system. However, the understanding of the dynamics based on equilibria alone is far from being complete, except for some rare cases, for instance, if a globally stable equilibrium exists.

The next natural step in the analysis of global dynamics is to check for periodic solutions and more complicated structures, like invariant circles or chaotic attractors. This step, however, is not trivial at all. A simple numerical approach to compute a large number of solutions starting from various initial conditions and to check whether these solutions escape to the infinity or settle down on some attractors may be satisfactory in some cases, but provides only limited amount of information, and is limited to attractors only. Moreover, the obtained results are only approximate, not mathematically rigorous. Even worse, this task becomes seriously more challenging if one is interested in tracing down the evolution of the invariant sets found as the parameters change and bifurcations occur, especially if some of the sets lose their stability.

An algorithmic method for solving the problem of determining global dynamics of a system of first-order ordinary difference equations in a mathematically reliable way was proposed in [1], and

is based upon rigorous numerical set-oriented computations. In this approach, global dynamics is decomposed into recurrent components (including fixed points or periodic solutions, independent of their stability type), and connections (connecting orbits) between these objects. Note that since all the recurrent dynamics in the system is captured in the recurrent components, these connections define a strict partial ordering on the recurrent components, and resemble connecting orbits between critical points in a gradient system on a smooth compact manifold. This strucutre is called a *Morse decomposition* of the phase space.

In order to determine the changes in the global dynamics that occur when the parameters are varying, a rigorous outer approximation of a Morse decomposition of the phase space is computed for small intervals into which the range of each parameter is subdivided. The computations are carried out at a predefined resolution in a bounded region of the phase space and for an *a priori* given subdivision of the parameters in some bounded ranges. Thanks to using interval arithmetic (see [6] for a comprehensive introduction to this subject) and carrying the computations out for the intervals of parameters, the results are rigorous, that is, all round-off errors that appear in floating-point arithmetic are accounted for, and the computed features are an outer estimate of the real dynamics given by the equations. Then classes of equivalent dynamics for different parameters are determined, and the parameter space is in this way subdivided into disjoint subsets experiencing different types of global dynamics.

The purpose of this paper is to provide a possibly simple introduction to this method, with an emphasis on how to appy it to a particular system of difference equations and how to interpret the results of the computations. The reader is referred to [1] for a formal introduction of this method and for all the technical details, and also to [4] for additional explanations and for an application of this method to a specific population model.

In Section 2, the terminology used in the remainder of the paper is introduced. In Section 3, a description of the input data for the computations is given. In Section 4, the information produced by the computational method is discussed. A sample application of this method is shown in Section 5. Finally, conclusion and some final remarks are gathered in Section 6.

2 Preliminaries

This section gathers several definitions and terminology used in the remainder of the paper.

Let $f\colon X \to X$ be a continuous map on a topological space X. A set $S \subset X$ is an *invariant set* with respect to f if $f(S) = S$. The *invariant part* of a set $N \subset X$ is $\mathrm{Inv}N := \bigcup\{S \subset N : f(S) = S\}$; it is an invariant set. The set N is called an *isolating neighborhood* if N is compact and its invariant part is contained in its interior, $\mathrm{Inv}N \subset \mathrm{int}N$. S is an *isolated invariant set* if $S = \mathrm{Inv}N$ for some isolating neighborhood N.

Any sequence $\{x_k\}_{k\in\mathbb{Z}}$ of points $x_k \in X$ such that $f(x_k) = x_{k+1}$ is called a *complete orbit* (or just an *orbit* for short). An analogous sequence indexed by the non-negative integers only is called a *positive orbit*.

A *Morse decomposition* (see [3]) of X with respect to f is a finite collection of disjoint isolated invariant sets (called *Morse sets*) $S_1, \ldots, S_p$ with a strict partial ordering $\prec$ on the index set $\{1, \ldots, p\}$ such that for every $x \in X \setminus (S_1 \cup \cdots \cup S_p)$ and for every complete orbit $\{\gamma_k\}_{k\in\mathbb{Z}}$ such that $\gamma_0 = x$ there exist indices $i \prec j$ such that $\gamma_k \to S_i$ as $k \to \infty$ and $\gamma_k \to S_j$ as $k \to -\infty$.

A family of isolating neighborhoods $N_1, \ldots, N_p$ with a strict partial ordering $\prec$ on the set of their indices is called a *numerical Morse decomposition* in a compact set $B \subset \mathbb{R}^n$ if the family

$\{S_i := \mathrm{Inv}N_i : i = 1, \ldots, p\}$ forms a Morse decomposition of $\mathrm{Inv}B$ with the ordering $\prec$. The sets N_i, $i = 1, \ldots, p$, are then called *numerical Morse sets*.

A [numerical] Morse decomposition can be schematically depicted as a directed graph whose vertices correspond to the [numerical] Morse sets and edges indicate possible connecting orbits between them. In order to simplify such a representation, the transitive reduction of this graph is plotted instead.

The *Conley index*, introduced by Conley [3] for flows, and generalized, e.g., by Mrozek [7] and Szymczak [12] to discrete semidynamical systems induced by continuous maps, is a topological invariant that provides information about each isolated invariant set $S_i := \mathrm{Inv}N_i$, based on algebraic-topological properties of N_i. This index takes into account the *exit set* of N, that is, the closure of $f(N) \setminus N$, and thus reflects the stability of what N contains. The knowledge of the Conley index of an isolating neighborhood N allows to draw conclusions on the invariant part of N. In particular, if the index of N is nontrivial then $\mathrm{Inv}N \neq \emptyset$. The index can also be used to prove the existence of periodic orbits or more complicated dynamics.

Since detailed introduction to the Conley index is beyond the scope of this paper and requires certain knowledge of algebraic topology, we refer the reader to [3, 7, 12] for more details on the Conley index, and to [5, 10] and references therein for discussion of some technical aspects of the computation of this index in the way in which it was implemented in the software referred to in this paper. Brief explanation on how to interpret the information on the Conley indices of isolating neighborhoods in the online presentation of the results can also be found in [1].

The Conley index, as well as the relation of $f(N)$ with respect to N, enable classifying the isolating neighborhoods on the basis of their stability. We say that an isolating neighborhood N is *attracting* if $f(N) \subset N$, that is, if the image of N is entirely contained in N. One can prove that then N contains a local attractor, which justifies this terminology. Otherwise, if the image of N is not fully contained in N, we say that N is *unstable*. If N has the Conley index of a hyperbolic fixed point or a hyperbolic periodic orbit with d-dimensional unstable manifold then we say that N *is of the type* of the corresponding point or orbit. For a typical system, it is likely that N indeed contains a periodic orbit of the expected period, but—since the Conley index is a purely topological tool and does not provide information about derivatives—the actual stability of such an orbit may be different, and the dynamics in N may turn out to be much more complicated than seen from outside (that is, from the perspective of the isolating neighborhood). If $N \subset \mathbb{R}^n$ is of the type of a fixed point or a periodic orbit with n-dimensional unstable manifold then we say that N is *repelling*.

A *rectangular set* is a product of compact intervals. Given a rectangular set

$$R = [a_1, a_1 + \delta_1] \times \cdots \times [a_n, a_n + \delta_n] \subset \mathbb{R}^n$$

and integer numbers $s_1, \ldots, s_n > 0$, we call the following set an $s_1 \times \cdots \times s_n$ *uniform rectangular grid in R*:

$$\mathcal{G}_{s_1,\ldots,s_n}(R) := \left\{ \prod_{i=1}^{n} [a_i + \frac{j_i}{s_i}\delta_i, a_i + \frac{j_i + 1}{s_i}\delta_i] : j_i \in \{0, \ldots, s_i - 1\}, i \in \{1, \ldots, n\} \right\}$$

The individual boxes in the grid are referred to by the n-tuples $(j_1, \ldots, j_n)$ for convenience. The n-tuple of integers $(s_1, \ldots, s_n)$ is called the *resolution* in R.

3 Input to the method

The computational method introduced in [1] can be applied to an m-parameter family of continuous maps on $\mathbb{R}^n$:

$$f\colon \mathbb{R}^n \times \mathbb{R}^m \ni (x,\lambda) \mapsto f_\lambda(x) \in \mathbb{R}^n$$

This family must be either given by an analytical formula, or at least by means of a method for computing a rigorous outer bound for $f(\bar{x},\bar{\lambda})$, where both $\bar{x}$ and $\bar{\lambda}$ are rectangular boxes, that is, products of (small) compact intevals.

The method requires that a rectangular set $\Lambda \subset \mathbb{R}^m$ that contains all the values of parameters of interest is given, as well as a resolution $(s_1,\ldots,s_m)$ in Λ.

Moreover, a rectangular set $B \subset \mathbb{R}^n$ that represents the area in the phase space on which the global dynamics is to be analyzed must be defined before the method can be applied, and a resolution $(d_1,\ldots,d_n)$ in B must be set up at which the computations are going to be carried out. In the current revision of the software, it is required that the numbers d_i are all equal, and are a positive power of 2.

Eventually, there are some additional settings of the computations that can be adjusted, but these are out of the scope of the paper, so the reader is referred to the source code of the software and the information provided in the configuration files for a sample system.

4 Output of the computations

In the computational method introduced in [1], for each parameter box $L \subset \Lambda$ in the $s_1 \times \cdots \times s_m$ uniform rectangular grid in Λ, a rigorous outer estimate of the map f_λ valid for all the parameters $\lambda \in L$ is automatically computed using interval arithmetic, on each of the boxes in the $d_1 \times \cdots \times d_n$ uniform rectangular grid in B. Then a numerical Morse decomposition is constructed in B using the computed common combinatorial represenation of f_λ for all $\lambda \in L$. Each numerical Morse set is built of boxes at the requested resolution in B. These numerical Morse decompositions constitute part of the output of the method, and are encoded as directed graphs, whose vertices contain information on the combinatorial Morse sets and edges represent the transitive reduction of the strict partial order relation $\prec$. Although it is possible to actually save the combinatorial Morse sets, this is not a feasible thing to do due to the tremendous size of such data. Therefore, the combinatorial Morse sets themselves are not considered to be part of the output of the computations, but rather intermediate temporary data. It is only brief information about the sets (the size and the Conley index, and optionally some additional features) which is collected, as well as the relation between the sets computed for adjacent parameter boxes, as discussed below.

The collected and further displayed data about each combinatorial Morse set consists of the consecutive number of the set, the number of boxes that form the set, and its Conley index. Although the former two pieces of information are self-explanatory, the latter one needs some explanation. The definition of the homological Conley index is based upon an index pair, and, roughly speaking, consists of the relative homology of the index pair, as well as the map induced in relative homology, further called the *index map*. This data may be represented as a finite sequence of finitely generated groups, e.g., $(\mathbb{Z},0,\mathbb{Z}^2)$, together with the images of homology generators by the index map, expressed as combinations of homology generators, at each level of gradation separately. Since the actual Conley index involves additional reduction (or otherwise it depends on the index pair instead of being an invariant of the isolated invariant set), and the reduced canonical

form seems to be very hard to compute, the non-zero eigenvalues of the index map may be used as a simplified and reliable invariant.

For each isolating neighborhood N constructed with the computational method, an index pair can be easily determined by taking N and the part of the forward image of N which sticks out of N, provided that this image and also its further image are both fully contained in the phase space B. Otherwise, it is unknown whether N is an isolating neighborhood in $\mathbb{R}^n$, and then the Conley index cannot be computed. This situation typically happens if N is too close to the boundary of B. For example, in the sample computations described in Section 5 this was precisely the case each time with the neighborhood of the origin.

For each pair of adjacent parameter boxes $L_1, L_2 \subset \Lambda$, the constructed numerical Morse decompositions are matched and a mathematically rigorous proof of continuation is made whenever possible. Such parameter boxes are gathered into equivalence classes, further called *continuation classes*. For each continuation class, the corresponding numerical Morse decomposition looks the same, even though the actual locations of the isolating neighborhoods in the phase space and their sizes may vary. The continuation classes are considered to be the most important part of the output produced by the computational method, because by analyzing the subdivision of the parameter space into continuation classes, one obtains a comprehensive overview of the dependence of the dynamics on parameters. The boxes in each class are saved by the software to a separate file for further processing.

In the case of a two-dimensional parameter space and a two-dimensional phase space, the results of the computations can be turned into an interactive presentation as a clickable continuation diagram, where each parameter box is linked to the corresponding graph depicting the computed combinatorial Morse decomposition, as well as a phase space portrait of the combinatorial Morse sets. Precisely this case is illustrated in Section 5 and at the website [8].

5 Sample application

In this section, an application of the above-described computational method to a sample two-dimensional map with two varying parameters is described.

Consider a two-stage discrete-time population model given by the following equations:

$$A_{n+1} = s_j J_n + s_a A_n$$
$$J_{n+1} = g(A_n)$$

where A_n denotes the number of adult individuals at time n, J_n denotes the number of juveniles at time n, and $g(x) = \alpha x e^{-\beta x}$ is the Ricker map (see [11]) with $\alpha > 1$ and $\beta > 0$. This is a variant of a system used for modeling baleen whales population in [2]. In order to simplify the presentation, α is set to a fixed value of e^4, for which it is known that the dynamics is not very complicated, but still not completely trivial. Since β turns out to be a scaling parameter, without loss of generality one can fix $\beta := \alpha/e$, so that the phase space is rescaled to $[0, 1]^2$. Due to numerical overestimates and the need to trace connecting orbits, however, one must take a considerably larger phase space for the actual computations. A quick computation at low resolution suggested that the choice of $B := [0, 4]^2$ would be appropriate to contain isolating neighborhoods for all the invariant sets, and would provide a large enough margin so that virtually no connecting orbits are missed.

Since the ranges for s_a and s_j valid for the model are both $[0, 1]$, but the value of 1 is highly questionable from the biological point of view, and indeed leads to weird behavior of solutions

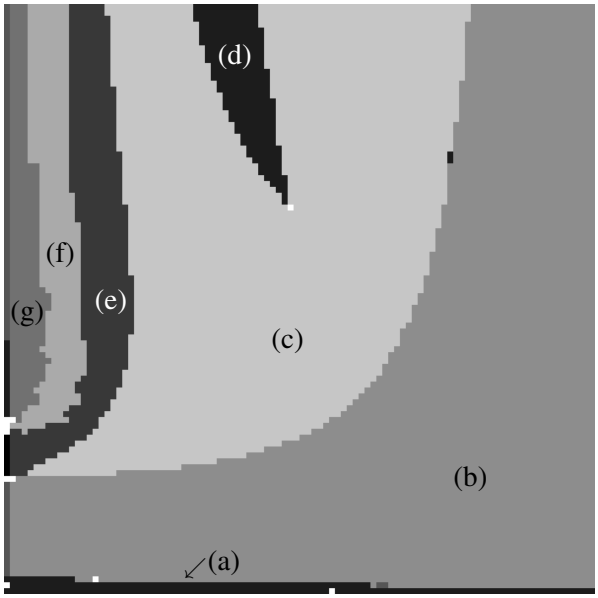

Figure 1: Continuation diagram computed for the sample system of difference equations (color online, see [8]). Each continuation class consisting of more than 1 element is indicated in some shade of gray (solid color online), with colors repeated for small classes that are at some distance from each other. The horizontal axis corresponds to $s_a \in [0, 0.9]$, and the vertical axis—to $s_j \in [0, 0.9]$. The labeled continuation classes are discussed in the text.

which may mess up with the interpretation of the results, for the purpose of this paper, both parameters are set to vary from 0 to 0.9, and thus the parameter space is set to $\Lambda := [0, 0.9]^2$.

The computations were conducted with the parameter space Λ subdivided uniformly into 100×100 boxes, and with B subdivided into 2048×2048 boxes. The choice of the resolutions was made on the basis of the cost of the computations, and it was checked empirically that with the chosen values, the memory usage did not exceed 350 MB per process, and the total CPU time was below 48 hours. The computations were done on a computer cluster, using a modified and updated version of the software referred to in [1]. An interactive illustration of the results of these computations is provided at [8]. Explanation and discussion of these results follows.

In Fig. 1, the continuation diagram computed for the system under consideration at the given resolution is shown. Some main continuation classes have been given labels and are briefly discussed below. Sample numerical Morse decompositions found in a selection of these classes are illustrated in Fig. 2. The reader is invited to explore the details at the interactive presentation provided at [8].

For parameters in the continuation class labeled (a), a single isolating neighborhood was found, and it is the neighborhood of the origin. Since it touches the boundary of the region B, its Conley index could not be computed.

For parameters in the continuation class (b), the neighborhood of the origin is unstable, and another attracting neighborhood at some distance from the origin appears. As the parameters approach the border between (b) and (c), the attracting neighborhood gets larger, and in the continuation class (c) it is split into an attracting ring that surrounds a small repelling isolating neighborhood inside (see Fig. 2, the fist picture).

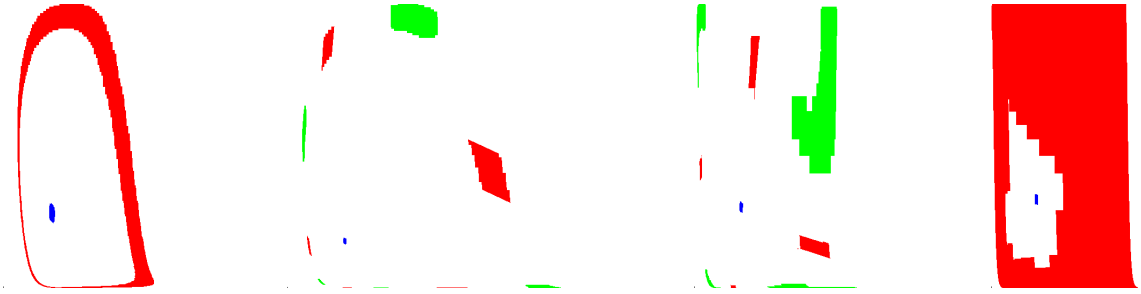

Figure 2: Sample numerical Morse decompositions in the phase space, taken for the parameters in the continuation classes (c), (d), (e), and (f), respectively, from left to right.

In the continuation class (d), the attracting ring observed in (c) is split into a combination of two isolating neighborhoods of the type of an attracting and saddle 5-periodic orbits (see Fig. 2, the second picture). In the continuation class (e), on the other hand, the attracting ring is split into a combination of two 4-periodic orbits (see Fig. 2, the third picture).

The phase space portrait in the continuation class (f) is very similar to the one observed in (c), except that the attracting ring is much thicker (see Fig. 2, the fourth picture). In the continuation class (g), the unstable isolating neighborhood and the surrounding attracting ring are no longer separated from each other, and a very large isolating neighborhood is observed in their place. This neighborhood joins the origin if one moves the parameters to the left from the continuation class (g) in the upper part of the diagram (higher values of s_j).

Finally, in the small thin continuation classes along the lower part of the left edge of the diagram, containing $s_a = 0$, synchronous solutions can be observed, in which one of the age groups is (nearly) empty, and the population oscilates between the "adults only" and "juveniles only" states.

6 Conclusion and final remarks

A rigorous numerical method for the determination of global dynamics of a system of first-order ordinary difference equations at an *a priori* given resolution was described. The method was illustrated with an application to a two-dimensional population model with two varying parameters. The wide applicability of this method is its greatest strength, as the assumptions on the system of difference equations for which it can be applied are very mild.

Although the cost of computations may be considerable for thorough analysis of a system, it was shown that already at relatively coarse resolutions one may be able to obtain interesting and meaningful results. Moreover, an important feature of this method is that the human effort that must be put into applying it to a particular system is very small, and boils down to the preparation of a small piece of code for the computation of the map, and for defining the rectangular phase space, parameter space, and resolutions in both.

The software for applying the method discussed in this paper is published at [9] and can be used to conduct comprehensive computations at multiple computers. An updated version of this software is available from the author upon request. A new implementation of the software is currently being developed as the *Conley-Morse-Database* project, and is expected to provide a more flexible and efficient means for the computations.

Acknowledgments

Participation of the author in the FDDE workshop as well as the preparation of this paper were partially supported from Departamento de Matemática Aplicada II of Universidade de Vigo, and also from Fundo Europeu de Desenvolvimento Regional (FEDER) through COMPETE—Programa Operacional Factores de Competitividade (POFC) and from the Portuguese National Funds through Fundação para a Ciência e a Tecnologia (FCT) in the framework of the research projects FCOMP-01-0124-FEDER-010645 (ref. FCT PTDC/MAT/098871/2008) and Est-C/MAT/UI0013/2011.

Moreover, the author thanks the Computer Science and Technology Center at Departamento de Informática of Universidade do Minho for providing their SeARCH cluster computing facilities.

References

[1] Z. Arai, W. Kalies, H. Kokubu, K. Mischaikow, H. Oka, P. Pilarczyk, *A database schema for the analysis of global dynamics of multiparameter systems,* SIAM J. Appl. Dyn. Syst. **8** (2009), 757–789.

[2] C.W. Clark, *A delayed recruitment model of population dynamics with an application to baleen whale populations,* J. Math. Biol. **3** (1976), 381–391.

[3] C. Conley, *Isolated invariant sets and the Morse index,* Amer. Math. Soc., Providence, RI, 1978.

[4] E. Liz, P. Pilarczyk, *Global dynamics in a stage-structured discrete-time population model with harvesting*, J. Theor. Biol., accepted.

[5] K. Mischaikow, M. Mrozek, P. Pilarczyk, *Graph approach to the computation of the homology of continuous maps,* Foundations of Computational Mathematics **5** (2005), 199–229.

[6] R.E. Moore, *Interval analysis,* Prentice–Hall, Inc., Englewood Cliffs, N.J., 1966.

[7] M. Mrozek, *Leray functor and cohomological Conley index for discrete dynamical systems,* Trans. Am. Math. Soc. **318** (1990), 149–178.

[8] P. Pilarczyk, *Set-oriented numerical analysis of a baleen whales population model. Results of computations,* `http://www.pawelpilarczyk.com/whales/` (2011).

[9] P. Pilarczyk, *Conley-Morse graphs computation software,* `http://www.pawelpilarczyk.com/cmgraphs/` (2009).

[10] P. Pilarczyk, K. Stolot, *Excision-preserving cubical approach to the algorithmic computation of the discrete Conley index,* Topology Appl. **155** (2008), 1149–1162.

[11] W.E. Ricker, *Stock and recruitment,* J. Fish. Res. Bd. Can. **11** (1954), 559–623.

[12] A. Szymczak, *The Conley index for discrete semidynamical systems,* Topology Appl. **66** (1995), 215–240.