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SHADOWING IS GENERIC—A CONTINUOUS MAP CASE

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Abstract. We prove that shadowing (the pseudo-orbit tracing property), periodic shadowing (tracing periodic pseudo-orbits with periodic real trajectories), and inverse shadowing with respect to certain families of methods (tracing all real orbits of the system with pseudo-orbits generated by arbitrary methods from these families) are all generic in the class of continuous maps and in the class of continuous onto maps on compact topological manifolds (with or without boundary) that admit a decomposition (including triangulable manifolds and manifolds with handlebody).

1. **Introduction.** Discrete-time (semi)dynamical systems are often investigated by means of numerical simulations, in which approximate (semi)trajectories are obtained in the iterated process of computing the image of a point by the map that generates the system. The (typically small) errors introduced at each step can accumulate, so the pseudo-orbits computed numerically may turn out to be very different from the real orbits of the system under investigation. We say that a map has the *pseudo-orbit tracing property*, also called *shadowing*, if arbitrarily close to the orbits computed numerically there exist real trajectories of the system, provided

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that the precision of the computations is high enough (obviously, the precision that is necessary depends on the required closeness); see Definition 2.3 below for the technical formulation of this feature. The shadowing property can provide a powerful tool for proving that the result of a numerical simulation describes dynamics that is indeed present in the system, and is not merely a side effect of errors accumulating during the computation. This is especially important if some interesting (complicated) trajectories are observed whose existence is not easy to prove analytically (see [16, 17] for examples of possible aplications). A subtle question is, in particular, whether a periodic pseudo-orbit encountered in a numerical simulation corresponds to a real periodic trajectory of the system. This is not guaranteed by shadowing alone; however, a variant of this property called *periodic shadowing* provides this feature; see Definition 2.3 for details.

Throughout the recent years, many researchers have worked on the subject of shadowing, so presently it is posible to devote entire monographs solely to this topic, and several tools for proving shadowing for specific dynamical systems have been developed (see [27, 28] and references therein).

Roughly speaking, shadowing guarantees that the numerically observed trajectories correspond to real orbits in the system under investigation. A question complementary in a sense is whether all trajectories present in the system can be actually observed using a numerical method that produces pseudo-orbits. Namely, we say that a map has the *inverse shadowing* property with respect to a given class of (numerical) methods (as in Definition 2.4) if for every method of sufficient precision, every orbit of the dynamical system is traced by a close pseudo-orbit generated by the method (and the precision that is necessary depends on the required closeness); see Definition 2.5 for details.

Taking the importance of shadowing into consideration, the question of whether this is a frequent property of dynamical systems or not, is of considerable interest. Unfortunately, it is impossible to give a single satisfactory answer to this question, since what is common in one class can be rare in another, and the class under consideration depends on a particular application. One of widely accepted methods for indicating that a property is typical is proving its genericity. The classical version of the shadowing lemma was proved in the 1970s independently by Anosov [1] and Bowen [4] in the context of diffeomorphisms (following the general trend of working in this context at that time), and supposedly this is the main reason for why the question of genericity of shadowing has been studied almost exclusively for invertible mappings.

The first result on the genericity of shadowing in the class of homeomorphisms was obtained for the unit circle [33], and was followed by the result for homeomorphisms on a compact manifold with dimension smaller than or equal 3 [26], and for homeomorphisms on any smooth compact manifold without boundary [30]. Recently, it has been proved that both shadowing and periodic shadowing are generic in the class of homeomorphisms on a compact smooth manifold, possibly with boundary [19, 21]. It is worth to note that the periodic shadowing property was introduced in [7] as a tool for proving existence of periodic orbits; another method for proving existence of periodic orbits might be a combination of shadowing with a certain form of expansivity, like in Bowen's classical decomposition theorem [2]; however, hyperbolic maps are not generic. It has also recently been proved that the shadowing property is generic in the class of $\mathbb{Z}^2$-actions on the interval [20] and that the inverse shadowing property is generic for homeomorphisms on a compact smooth manifold, possibly with boundary [12, 23].

These results provide considerable amount of information about genericity of shadowing in the invertible case. However, it is not clear whether these maps are representative enough in the entire class of semidynamical systems. Definitely, in dimension one they are not, because all the maps interesting from the dynamical point of view (that is, those with positive topological entropy) are non-invertible (because positive topological entropy is generated by horseshoes [24]). Unfortunately, there are very few results on genericity of shadowing in dimension one in the non-invertible setting. In particular, it is known that shadowing is generic for continuous maps on the unit interval and on the unit circle [25]. Moreover, almost every tent map has the shadowing property [9], which is also true for a larger class of piecewise linear maps [6, 9].

As far as we are aware, there is no proof in the literature that shadowing is generic in the class of (possibly non-invertible) continuous surjections or, more widely, continuous maps in dimension 2 or higher. The main aim of our work is to contribute towards filling this gap by proving the following theorems:

Theorem 1.1. *Let M be a compact topological manifold that admits a decomposition (as in Definition 2.1). Then the shadowing property is generic in the class $C(M)$ of continuous maps on M, as well as in the class $S(M)$ of continuous onto maps on M.*

Theorem 1.2. *Let M be a compact topological manifold that admits a decomposition (as in Definition 2.1). Then the periodic shadowing property is generic in the class $C(M)$ of continuous maps on M, as well as in the class $S(M)$ of continuous onto maps on M.*

Theorem 1.3. *Let M be a compact topological manifold that admits a decomposition (as in Definition 2.1). Then the inverse shadowing property with respect to the family $\mathcal{T}_S$ as well as with respect to the family $\mathcal{T}_H$ (see Definition 2.6) is generic in the class $C(M)$ of continuous maps on M, as well as in the class $S(M)$ of continuous onto maps on M.*

In Section 2, we introduce the notation and set up several definitions and assumptions used further in the paper. We also recall necessary background material. In Section 3, we prove Theorems 1.1, 1.2, and 1.3 through a series of lemmas and propositions.

2. Preliminaries.

We begin with discussing the assumptions on the manifold in Section 2.1. Then we recall the necessary definitions related to shadowing in Section 2.2 and to inverse shadowing in Section 2.3. We conclude with recalling the definition of a covering relation and providing some relevant results in Section 2.4.

2.1. Manifolds and decompositions.

Denote the set of nonnegative integers by $\mathbb{N}$. Denote the unit open ball in $\mathbb{R}^k$ by $\mathbb{I}^k$.

Let M be a compact k-dimensional ($k \geq 1$) topological manifold (with or without boundary). Since M is a second countable Hausdorff regular space, it is metrizable, so without loss of generality it can be assumed that M is endowed with a metric d compatible with the topology on M. For $x \in M$ and $r > 0$, let $B(x, r)$ denote the closed ball of radius r centered at x. Let

$$\mathcal{A} = \{(H_1, \phi_1), \ldots, (H_l, \phi_l)\}$$

be a finite atlas on M, where $\phi_i\colon H_i \to \mathbb{I}^k$ if M is a manifold without boundary and $\phi_i\colon \overline{H_i} \to \overline{\mathbb{I}^k}$ if M is a manifold with boundary. By replacing the sets H_i with smaller ones whenever necessary, it can be assumed without loss of generality that each ϕ_i can be extended to an injective continuous map $\phi_i\colon \overline{H_i} \to \overline{\mathbb{I}^k}$ if M is a manifold without boundary.

Definition 2.1. Let M be a compact k-dimensional ($k \geq 1$) topological manifold. A finite family $\mathcal{S}$ of pairwise disjoint open subsets of M is called a *decomposition* of M if $M = \bigcup_{U\in\mathcal{S}} \overline{U}$, and each $\overline{U}$ is homeomorphic to a closed ball in $\mathbb{R}^k$. If there exists a decomposition of M then we say that M *admits a decomposition*.

Note that triangulable manifolds or manifolds with handlebody admit a decomposition. It is known that all compact manifolds of dimension at most 3 are triangulable [13, 3]. Moreover, all compact manifolds of dimension at least 6 possess a handlebody [14]. On the other hand, an example of a 4-dimensional compact manifold is known which does not admit triangulation [11]. It can also be proved that for each $D > 4$, there exists a closed manifold of dimension D which cannot be triangulated [15]. However, some conditions sufficient for triangulation are known. For example, all smooth manifolds are triangulable [5, 32].

If $\mathcal{S}$ is a decomposition of M then it is easy to see that by applying appropriate consecutive subdivisions of the sets in $\mathcal{S}$, it is possible to construct a sequence $\{\mathcal{S}_n\}_{n=0}^{\infty}$ of decompositions of M satisfying the following conditions:

(i) $\mathcal{S}_0 = \mathcal{S}$,
(ii) $\operatorname{diam}(\mathcal{S}_n) := \max\{\operatorname{diam} U \mid U \in \mathcal{S}_n\} < \frac{1}{n}$ for $n \geq 1$,
(iii) for every $n \in \mathbb{N}$, each set $U \in \mathcal{S}_{n+1}$ is contained in some set $V \in \mathcal{S}_n$.

Moreover, by replacing $\mathcal{S}$ with $\mathcal{S}_n$ for a suitably large n, one can make $\operatorname{diam}\mathcal{S}$ smaller than the Lebesgue number of the covering of M by the charts $H_1, \dots, H_l$. This can be understood as some kind of consistency of $\mathcal{S}$ with $\mathcal{A}$, because it implies, in particular, the fact that the closure of every set $U \in \mathcal{S}$ is contained in some chart H_i, $i \in \{1, \dots, l\}$.

From now on we shall assume that the manifold M admits a decomposition $\mathcal{S}$ with $\operatorname{diam}\mathcal{S}$ smaller than the Lebesgue number of the covering of M by the charts $H_1, \dots, H_l$, and that $\{\mathcal{S}_n\}_{n=0}^{\infty}$ is a sequence satisfying the conditions (i)–(iii).

Since our reasoning is going to be conducted locally in the charts of the atlas $\mathcal{A}$ or in the sets of the decompositions $\mathcal{S}_n$, for simplicity of notation, we are going to describe it as if it was done in (subsets of) $\mathbb{R}^k$.

2.2. **Shadowing.** A homeomorphism $f\colon M \to M$ induces a *dynamical system* on M whose *trajectories* (also called *orbits*) are $\{f^n(x)\}_{k\in\mathbb{Z}}$, where $x \in M$. A continuous map $f\colon M \to M$ induces a *semidynamical system* on M with *semitrajectories* $\{f^n(x)\}_{k\in\mathbb{N}}$. Since we are going to work exclusively with continuous maps that are not necessarily invertible, in order to simplify the terminology, from now on we are going to skip the prefix "semi-".

Let $C(M)$ and $S(M)$ denote the spaces of all continuous maps on M and continuous surjections on M, respectively, both equipped with the complete metric

$$\rho_0(f, g) := \max\{d(f(x), g(x)) \mid x \in M\}.$$

Definition 2.2. Let $f\colon M \to M$. Let $\delta > 0$. A sequence $\{y_n\}_{n\in\mathbb{N}} \subset M$ is called a δ*-pseudo-orbit* of f if

$$d(f(y_n), y_{n+1}) < \delta \text{ for every } n \in \mathbb{N}.$$

Moreover, if there exists $N > 0$ such that $y_{n+N} = y_n$ for all $n \in \mathbb{N}$ then this δ-pseudo-orbit is called *periodic*.

Definition 2.3. A map $f\colon M \to M$ has the *shadowing property* (or: *periodic shadowing property*) if for every $\varepsilon > 0$ there exists $\delta > 0$ satisfying the following condition: for every δ-pseudo-orbit (or: periodic δ-pseudo-orbit, respectively) $\mathrm{y} = \{y_n\}_{n\in\mathbb{N}}$ there exists an orbit (or: a periodic orbit, respectively) $\mathrm{x} = \{x_n\}_{n\in\mathbb{N}} = \{f^n(x_0)\}_{n\in\mathbb{N}}$, which ε-*traces* y, that is,

$$d(x_n, y_n) \le \varepsilon \text{ for every } n \in \mathbb{N}.$$

We would like to remark that the notion of a δ-pseudo-orbit and that of ε-tracing both do depend on the metric d chosen at the manifold M in Section 2.1, and thus it might potentially be the case that some maps may have a shadowing-related property (like shadowing, periodic shadowing, or inverse shadowing discussed in Section 2.3) with respect to one metric and lack this property with respect to another equivalent but not strongly equivalent metric. However, in what follows, we prove that there exist residual sets of maps which have each of the considered properties, that is, we prove that each of these properties is satisfied for a countable intersection of open and dense sets in the complete metric space of continuous maps on M or continuous onto maps on M with the C^0 topology. And this result is independent of the choice of a metric on M compatible with its topology.

2.3. **Inverse shadowing.** Let $M^{\mathbb{N}}$ be the compact space of the one-sided sequences $\{x_n\}_{n\in\mathbb{N}} \subset M$, endowed with the Tikhonov product topology.

Definition 2.4. Given $\delta > 0$, a δ-*method* for a map $f\colon M \to M$ is a mapping $\chi\colon M \to M^{\mathbb{N}}$ such that for every $x \in M$, the sequence $\chi(x)$ is a δ-pseudo-orbit of f with $\chi(x)_0 = x$. We say that a family $\mathcal{T} = \{\mathcal{T}_\delta \mid \delta > 0\}$ of classes $\mathcal{T}_\delta$ of δ-methods is *complete* if $\mathcal{T}_\delta \neq \emptyset$ for every $\delta > 0$.

Definition 2.5. Let $f\colon M \to M$ and let $\mathcal{T} = \{\mathcal{T}_\delta \mid \delta > 0\}$ be a complete family of classes $\mathcal{T}_\delta$ of δ-methods for f. We say that f has the *inverse shadowing* property with respect to $\mathcal{T}$ (called the $\mathcal{T}$-*inverse shadowing* property for short) if for every $\varepsilon > 0$ there exists $\delta > 0$ satisfying the following condition: for every $\chi \in \mathcal{T}_\delta$ and for every orbit $\mathrm{x} = \{x_n\}_{n\in\mathbb{N}}$ there exists a point $y \in M$ such that the δ-pseudo-orbit $\chi(y)$ ε-traces x.

Taking into account potential applications, one is tempted to look for as large a complete family as possible. However, this family cannot be too large, because, for example, the family of classes of all the possible δ-methods for f is of limited interest, since there is no structurally stable system satisfying the inverse shadowing property with respect to this family [8]. Following [18, 29], in the present paper we limit our attention to the following complete families of classes of continuous δ-methods:

Definition 2.6. For every $\delta > 0$, let $\mathcal{T}_{F,\delta}$ denote the class of all the possible δ-methods for f. Define the following classes:

$$\mathcal{T}_{H,\delta} := \{\chi \in \mathcal{T}_{F,\delta} \mid \text{there exists a homeomorphism } \psi\colon M \to M \text{ such that } \chi(x)_{n+1} = \psi(\chi(x)_n) \text{ for all } x \in M, n \in \mathbb{N}\},$$

$$\mathcal{T}_{S,\delta} := \{\chi \in \mathcal{T}_{F,\delta} \mid \text{there exists a family of continuous maps } \psi_n\colon M \to M \text{ such that } \rho_0(\psi_n, f) < \delta \text{ and } \chi(x)_{n+1} = \psi_n(\chi(x)_n) \text{ for all } x \in M, n \in \mathbb{N}\},$$

$$\mathcal{T}_{C,\delta} := \{\chi \in \mathcal{T}_{F,\delta} \mid \text{there exists a family of continuous maps } \psi_n\colon M \to M \text{ such that } \chi(x)_n = \psi_n(x) \text{ for all } x \in M, n \in \mathbb{N}\},$$

and define the following families: $\mathcal{T}_H := \{\mathcal{T}_{H,\delta} \mid \delta > 0\}$, $\mathcal{T}_S := \{\mathcal{T}_{S,\delta} \mid \delta > 0\}$, $\mathcal{T}_C := \{\mathcal{T}_{C,\delta} \mid \delta > 0\}$.

Unlike in the invertible case [29], where $\mathcal{T}_{H,\delta} \subsetneq \mathcal{T}_{S,\delta} \cap \mathcal{T}_{C,\delta}$, but $\mathcal{T}_{S,\delta} \nsubseteq \mathcal{T}_{C,\delta}$ and $\mathcal{T}_{C,\delta} \nsubseteq \mathcal{T}_{S,\delta}$, here we have the following result:

Observation 2.7. *For all $\delta > 0$, $\mathcal{T}_{H,\delta} \subsetneq \mathcal{T}_{S,\delta} \subsetneq \mathcal{T}_{C,\delta}$.*

Proof. The proper inclusion $\mathcal{T}_{H,\delta} \subsetneq \mathcal{T}_{S,\delta}$ is obvious, and the inclusion $\mathcal{T}_{S,\delta} \subsetneq \mathcal{T}_{C,\delta}$ follows from the following argument. Take $\chi \in \mathcal{T}_{S,\delta}$. Take the corresponding family $\{\psi_n\}$. Then the family of continuous maps $\{\tilde{\psi}_n\}$ defined by $\tilde{\psi}_n := \psi_{n-1} \circ \cdots \circ \psi_0$ provides in $\mathcal{T}_{C,\delta}$ the same δ-pseudo-orbits as the ones given in $\mathcal{T}_{S,\delta}$ by $\{\psi_n\}$. To see that the inclusion is proper, note that whenever $\chi(x)_N = \chi(y)_N$ for some $N \geq 0$ in $\mathcal{T}_{S,\delta}$ then $\chi(x)_n = \chi(y)_n$ for all $n \geq N$, which need not be the case in $\mathcal{T}_{C,\delta}$. □

We would like to point out the fact that in the invertible case (homeomorphisms) the problem of C^0 genericity of $\mathcal{T}_H$-inverse shadowing has already been investigated [12], as well as C^0 genericity of $\mathcal{T}_S$-inverse shadowing [23], and C^0 genericity of $\mathcal{T}_C$-inverse shadowing in the case of $\dim M \leq 3$ [22]. However, to our best knowledge, there are no results concerning C^0 genericity of any kind of inverse shadowing in the noninvertible case (continuous maps), nor any results concerning $\mathcal{T}_C$-inverse shadowing without the restriction on the dimension of the manifold (in both invertible and noninvertible cases).

2.4. **Covering relations.** In the proofs related to periodic shadowing, we shall need a special (simplified) version of covering relations introduced by Zgliczyński and Gidea [34]. We recall the necessary terminology below, prove a simple lemma, and quote two theorems from [34] that are going to be used in the sequel.

Let D^n denote the closed unit ball in $\mathbb{R}^n$, and let $S^{n-1} := \partial D^n$ denote its boundary, the unit sphere in $\mathbb{R}^n$. Let N, L be closed balls in $\mathbb{R}^n$, let $f\colon N \to \mathbb{R}^n$ be a continuous map, and let w be a nonzero integer. Define $f_c := (c_L \circ f \circ c_N^{-1})|_{D^n}\colon D^n \to \mathbb{R}^n$, where $c_N, c_L\colon \mathbb{R}^n \to \mathbb{R}^n$ are some homeomorphisms with $c_N(N) = D^n$ and $c_L(L) = D^n$. We say that N *f-covers L with degree w*, which we denote as $N \overset{f,w}{\Longrightarrow} L$, if $f_c(\partial D^n) \subset \mathbb{R}^n \setminus D^n$ and the local Brower degree $\deg(f_c, D^n, 0)$ of the map f_c in D^n at 0 equals w (see [34, Appendix A] for a summary of properties of the local Brouwer degree). Note that these conditions do not depend on the choice of c_L and c_N.

Lemma 2.8. *Let N and L be closed balls in $\mathbb{R}^n$, and let $f\colon N \to \mathbb{R}^n$ be a homeomorphism onto its image. If $L \subset \operatorname{int} f(N)$ then $N \overset{f,w}{\Longrightarrow} L$ for some nonzero integer w.*

Proof. For simplicity of notation, assume that $N = L = D^n$ and $f_c = f$. If $n = 1$ then $f(\partial N)$ consists of two endpoints of the segment $f(N)$ containing the segment L in its interior, and thus obviously $0 \notin f(\partial N)$. If $n \geq 2$ then by Jordan-Brouwer separation theorem, $\mathbb{R}^n \setminus f(\partial N)$ consists of two connected components, a bounded and an unbounded one, with $f(\partial N)$ being their common boundary. Obviously, $f(\operatorname{int} N)$ is mapped to the bounded component, which therefore contains L. In particular, $0 \notin f(\partial N)$ also if $n \geq 2$. This implies that the local Brouwer degree of f in N at 0 is well defined. The fact that f is a homeomorphism onto its image implies that this degree is nonzero (which follows, for instance, from the multiplication property of the degree applied to $\mathrm{id} = f \circ f^{-1}$). This concludes the proof. □

Theorem 2.9 (see [34], Theorem 13)**.** *If $N \overset{f,w}{\Longrightarrow} L$, then there exists $\varepsilon > 0$ such that if $\rho_0(f,g) < \varepsilon$ then $N \overset{g,w}{\Longrightarrow} L$.*

Theorem 2.10 (see [34], Theorem 9)**.** *If $N_0 \overset{f_1,w_1}{\Longrightarrow} N_1 \overset{f_2,w_2}{\Longrightarrow} \dots \overset{f_k,w_k}{\Longrightarrow} N_k$, then there exists $x \in \operatorname{int} N_0$, such that $(f_i \circ f_{i-1} \circ \dots \circ f_1)(x) \in \operatorname{int} N_i$ for all $i \in \{1,\dots,k\}$. Moreover, if $N_0 = N_k$ then x can be chosen in such a way that $(f_k \circ f_{k-1} \circ \dots \circ f_1)(x) = x$.*

3. **Proof of genericity of shadowing, periodic shadowing, and inverse shadowing.** In order to prove Theorems 1.1, 1.2 and 1.3, we proceed as follows. First, for each $n \in \mathbb{N}$, we define the sets $\mathcal{A}_n$, $\bar{\mathcal{A}}_n$ and $\hat{\mathcal{A}}_n$ of continuous maps on M which satisfy certain conditions. The definitions are set up in such a way that $\hat{\mathcal{A}}_n \subset \bar{\mathcal{A}}_n \subset \mathcal{A}_n$. Then we prove that all $\mathcal{A}_n$ and $\bar{\mathcal{A}}_n$ are open in $C(M)$. (Note that we do not prove that any $\hat{\mathcal{A}}_n$ is open; the role of these sets is different from that of $\mathcal{A}_n$ and $\bar{\mathcal{A}}_n$.) Next, we define the sets $\mathcal{B}_n$, $\bar{\mathcal{B}}_n$ and $\hat{\mathcal{B}}_n$ as the infinite unions of all the corresponding sets $\mathcal{A}_m$, $\bar{\mathcal{A}}_m$ and $\hat{\mathcal{A}}_m$, respectively, with indices starting at n. Then we prove that each of the sets $\hat{\mathcal{B}}_n$ is dense in $C(M)$, and also that its intersection with $S(M)$ is dense in $S(M)$. By the inclusion $\hat{\mathcal{A}}_n \subset \bar{\mathcal{A}}_n \subset \mathcal{A}_n$, we infer the density of the sets $\mathcal{B}_n$ and $\bar{\mathcal{B}}_n$. Eventually, we prove that each $f \in \mathcal{A}_n$ satisfies the shadowing property with $\varepsilon = \frac{1}{n}$, each $f \in \bar{\mathcal{A}}_n$ satisfies the periodic shadowing property with $\varepsilon = \frac{1}{n}$, and each $f \in \mathcal{A}_n$ (sic!) satisfies the inverse shadowing property with $\varepsilon = \frac{1}{n}$. All these facts taken together will yield the desired results.

3.1. **Definition of the sets $\mathcal{A}_n$, $\bar{\mathcal{A}}_n$ and $\hat{\mathcal{A}}_n$.**

Definition 3.1. We say that a family of sets $\mathcal{Q}$ is *inscribed* into another family of sets $\mathcal{R}$ if $\mathcal{Q}$ is a disjoint union of nonempty subfamilies $\mathcal{Q}_R$, indexed by elements of $\mathcal{R}$, such that each $Q \in \mathcal{Q}_R$ is contained in R.

Definition 3.2. Given $\delta > 0$, we say that a map $f \in C(M)$ is *δ-compatible* with a family $\mathcal{Q}$ of subsets of M inscribed into another family $\mathcal{R}$ of subsets of M if for any $U, V \in \mathcal{R}$, either $f(\overline{U}) \cap \overline{V} = \emptyset$, or for some $B \in \mathcal{Q}_U$ the following holds:

$$\bigcap \{g(B) \mid g \in C(M), \rho_0(f,g) < \delta\} \supset \bigcup \mathcal{Q}_V.$$

(In other words, there exists $B \in \mathcal{Q}_U$ such that for all $D \in \mathcal{Q}_V$ and for all $g \in C(M)$ such that $\rho_0(f,g) < \delta$, we have $g(B) \supset D$.)

If additionally $B \overset{f,w_D}{\Longrightarrow} D$ for some $B \in \mathcal{Q}_U$ and each $D \in \mathcal{Q}_V$ with $w_D \neq 0$ then we say that f is *strongly δ-compatible with $\mathcal{Q}$ and $\mathcal{R}$*.

For all $n \in \mathbb{N}$, define the sets $\mathcal{A}_n$, $\bar{\mathcal{A}}_n$ and $\hat{\mathcal{A}}_n$ in the following way.

Let $\mathcal{A}_n \subset C(M)$ be the set of all $f \in C(M)$ for which there exists a finite family B_n of pairwise disjoint closed balls in M such that B_n is inscribed into $\mathcal{S}_n$ and f is δ-compatible with B_n and $\mathcal{S}_n$ for some $\delta > 0$.

Let $\bar{\mathcal{A}}_n \subset C(M)$ be the set of all $f \in C(M)$ for which there exists a finite family B_n of pairwise disjoint closed balls in M such that B_n is inscribed into $\mathcal{S}_n$ and f is strongly δ-compatible with B_n and $\mathcal{S}_n$ for some $\delta > 0$.

Finally, let $\hat{\mathcal{A}}_n \subset C(M)$ be the set of all $f \in C(M)$ for which there exists a finite family B_n of pairwise disjoint closed balls in M such that B_n is inscribed into $\mathcal{S}_n$ and f is δ-compatible with B_n and $\mathcal{S}_n$ for some $\delta > 0$, and additionally $f|_B \colon B \to f(B)$ is a homeomorphism for each $B \in B_n$.

As an immediate consequence of Lemma 2.8 and the above definitions, we have the following result.

Observation 3.3. *For all $n \in \mathbb{N}$,*

$$\hat{\mathcal{A}}_n \subset \bar{\mathcal{A}}_n \subset \mathcal{A}_n.$$

3.2. **Openness of the sets $\mathcal{A}_n$ and $\bar{\mathcal{A}}_n$.**

Proposition 3.4. *The set $\mathcal{A}_n$ is open in $C(M)$. As a consequence, $\mathcal{A}_n \cap S(M)$ is open in $S(M)$.*

Proof. Let $f \in \mathcal{A}_n$ and let B_n and δ be provided for f by the definition of $\mathcal{A}_n$. Since $\mathcal{S}_n$ is a finite set, there exists a positive $\gamma < \delta$ such that for any two sets $U, V \in \mathcal{S}_n$ for which $f(\overline{U}) \cap \overline{V} = \emptyset$ we have

$$\operatorname{dist}(f(\overline{U}), \overline{V}) := \min\{d(x,y) \mid x \in f(\overline{U}), y \in \overline{V}\} \geq \gamma.$$

Consider $h \in C(M)$ with $\rho_0(f,h) < \min\{\frac{\gamma}{2}, \frac{\delta}{2}\}$ and take $U, V \in \mathcal{S}_n$ for which $h(\overline{U}) \cap \overline{V} \neq \emptyset$. Then $f(\overline{U}) \cap \overline{V} \neq \emptyset$, because otherwise

$$\operatorname{dist}(h(\overline{U}), \overline{V}) \geq \operatorname{dist}(f(\overline{U}), \overline{V}) - \rho_0(f,h) \geq \tfrac{\gamma}{2} > 0,$$

which would be a contradiction. Moreover, by the triangle inequality,

$$\rho_0(f,g) \leq \rho_0(f,h) + \rho_0(h,g)$$

for any $g \in C(M)$, so

$$\{g \in C(M) \mid \rho_0(h,g) < \tfrac{\delta}{2}\} \subset \{g \in C(M) \mid \rho_0(f,g) < \delta\}.$$

Hence it is easy to see that h is $\frac{\delta}{2}$-compatible with B_n and $\mathcal{S}_n$. This completes the proof. □

Proposition 3.5. *The set $\bar{\mathcal{A}}_n$ is open in $C(M)$. As a consequence, $\bar{\mathcal{A}}_n \cap S(M)$ is open in $S(M)$.*

Proof. This follows from Theorem 2.9 and Observation 3.3. □

3.3. **Definition of the sets $\mathcal{B}_n$, $\bar{\mathcal{B}}_n$ and $\hat{\mathcal{B}}_n$, and proof of their density.** For each $n \in \mathbb{N}$, define the sets $\mathcal{B}_n \subset C(M)$, $\hat{\mathcal{B}}_n \subset C(M)$ and $\bar{\mathcal{B}}_n \subset C(M)$ as follows:

$$\mathcal{B}_n := \bigcup_{m=n}^{\infty} \mathcal{A}_m, \qquad \bar{\mathcal{B}}_n := \bigcup_{m=n}^{\infty} \bar{\mathcal{A}}_m, \qquad \hat{\mathcal{B}}_n := \bigcup_{m=n}^{\infty} \hat{\mathcal{A}}_m.$$

The following is an immediate consequence of Propositions 3.4 and 3.5.

Observation 3.6. *The sets $\mathcal{B}_n$ and $\bar{\mathcal{B}}_n$ are open in $C(M)$ for all $n \in \mathbb{N}$. In particular, $\mathcal{B}_n \cap S(M)$ and $\bar{\mathcal{B}}_n \cap S(M)$ are open in $S(M)$ for all $n \in \mathbb{N}$.*

For the purpose of proving the density of $\hat{\mathcal{B}}_n$, we shall need the following lemmas. The first one is a version of Tietze extension theorem, which we repeat after [10, Theorem 4.1].

Lemma 3.7. *Let X be an arbitrary metric space, let A be a closed subset of X, let L be a locally convex linear space, and let $f \colon A \to L$ be a continuous map. Then there exists an extension $F \colon X \to L$ of f (that is, a continuous map identical to f on A), such that the image of F is contained in the convex hull of $f(A)$.*

The second lemma provides some kind of stability of open sets being covered, with respect to small perturbations of the map. This simplified version of [31, Lemma 2.3] provides a sufficient tool for our purposes.

Lemma 3.8. *Let A be a compact subset of E, and let $f\colon A \to F$ be a continuous map, where E and F are finite-dimensional normed spaces. Let B be a closed subset of A such that $f|_B\colon B \to f(B)$ is a homeomorphism. Then for any closed set $D \subset \operatorname{int} f(B)$ there exists $\varepsilon > 0$ such that for every continuous map $g\colon A \to F$, if g satisfies the condition*

$$\rho_0(f,g) := \max\{d(f(x), g(x)) \mid x \in A\} < \varepsilon,$$

then $D \subset g(B)$.

Finally, we prove the following result which is a key ingredient in the proof of density of $\hat{\mathcal{B}}_n$.

Lemma 3.9. *Let $f \in C(M)$ and $n \in \mathbb{N}$. Then for any $\varepsilon > 0$ there exists a map $g \in C(M)$ such that $\rho_0(f,g) < \varepsilon$, for every $U, V \in \mathcal{S}_n$ we have*

$$f(\overline{U}) \cap \overline{V} \neq \emptyset \iff g(\overline{U}) \cap \overline{V} \neq \emptyset,$$

and there exists a family of pairwise disjoint closed balls $\mathcal{K} = \{K_{U,V} \mid U, V \in \mathcal{S}_n$ and $f(\overline{U}) \cap \overline{V} \neq \emptyset\}$ such that $K_{U,V} \subset U$, $g(K_{U,V}) \subset V$, and $f(M) \subset g(M \setminus \bigcup \mathcal{K})$. In particular, if $f \in S(M)$ then $g \in S(M)$.

Proof. Take any $\varepsilon > 0$. For each $U \in \mathcal{S}_n$, consider the nonempty family

$$\mathcal{S}_{n,U} := \{V \in \mathcal{S}_n \mid f(\overline{U}) \cap \overline{V} \neq \emptyset\}$$

and the compact set

$$W_U := \bigcup \{\overline{V} \mid V \in \mathcal{S} \setminus \mathcal{S}_{n,U}\}.$$

Define

$$\gamma := \min\{\operatorname{dist}(f(\overline{U}), W_U) \mid U \in \mathcal{S}_n\} > 0$$

and take a $\delta > 0$ smaller than both ε and γ, and also smaller than the Lebesgue number of the covering of M by the seleced charts (see Section 2.1) to make sure that every set whose diameter does not exceed δ lies entirely in some element of the atlas $\mathcal{A}$ of M, and thus can be considered a subset of $\mathbb{R}^k$ up to a homeomorphism.

Because of the fact that $f(\overline{U}) \subset \overline{f(U)}$ for each $U \in \mathcal{S}_n$, and that each $f(\overline{U})$ is compact, for each $U \in \mathcal{S}_n$ there exists a finite set of points $\{x_1^U, \ldots, x_{l_U}^U\} \subset U$ satisfying

$$f(\overline{U}) \subset \bigcup_{i=1}^{l_U} B(f(x_i^U), \tfrac{\delta}{4}). \tag{1}$$

For each $U \in \mathcal{S}_n$ and $V \in \mathcal{S}_{n,U}$, there exists $x_{U,V}^0 \in \overline{U}$ such that $y_{U,V}^0 = f(x_{U,V}^0) \in \overline{V}$. Let $D_{U,V}$ be the closed ball of radius $\frac{\delta}{2}$ centered at $y_{U,V}^0$. Note that $D_{U,V} \cap V \neq \emptyset$ and $\operatorname{diam} D_{U,V} \leq \delta$.

For each $U \in \mathcal{S}_n$ and $V \in \mathcal{S}_{n,U}$, let $B_{U,V}^0$ be a closed ball centered at $x_{U,V}^0$ small enough that $f(B_{U,V}^0) \subset D_{U,V}$. Let $B_{U,V}^1$ be a (possibly smaller) closed ball contained in $B_{U,V}^0$ such that $B_{U,V}^1 \subset U$ and $B_{U,V}^1 \cap \{x_1^U, \ldots, x_{l_U}^U\} = \emptyset$. Let $\{B_{U,V} \mid U \in \mathcal{S}_n,\ V \in \mathcal{S}_{m,U}\}$ be a collection of (even smaller) closed balls that are mutually disjoint and contained in the corresponding balls $B_{U,V}^1$. Note that obviously $B_{U,V} \subset U$ and $f(B_{U,V}) \subset D_{U,V}$ for all $U \in \mathcal{S}_n$ and $V \in \mathcal{S}_{n,U}$.

Take a constant $\alpha > 0$ small enough that for each $U \in \mathcal{S}_n$, the balls $B(x_1^U, \alpha), \ldots, B(x_{l_U}^U, \alpha)$ are mutually disjoint and contained in U, and $f(B(x_i^U, \alpha)) \subset B(f(x_i^U), \frac{\delta}{2})$ for all $i \in \{1, \ldots, l_U\}$, and also that for each $V \in \mathcal{S}_{n,U}$,

$$B(x_i^U, \alpha) \cap B_{U,V} = \emptyset. \tag{2}$$

For each $U \in \mathcal{S}_n$ and $V \in \mathcal{S}_{n,U}$, choose any $x_{U,V} \in B_{U,V}$ and $y_{U,V} \in D_{U,V} \cap V$. By Lemma 3.7, there exist continuous maps $g_{U,V} \colon B_{U,V} \to D_{U,V}$ satisfying

$$\begin{aligned} g_{U,V}(x_{U,V}) &= y_{U,V}, \\ g_{U,V}|_{\partial B_{U,V}} &= f|_{\partial B_{U,V}}. \end{aligned}$$

For each $U \in \mathcal{S}_n$ and $i \in \{1, \ldots, l_U\}$, consider an arbitrary continuous onto map

$$\tilde{g}_{U,i} \colon B(x_i^U, \tfrac{\alpha}{2}) \to B(f(x_i^U), \tfrac{\delta}{4}). \tag{3}$$

Again by Lemma 3.7, there exists a continuous map $g_{U,i} \colon B(x_i^U, \alpha) \to B(f(x_i^U), \frac{\delta}{2})$ satisfying

$$\begin{aligned} g_{U,i}|_{B(x_i^U, \frac{\alpha}{2})} &= \tilde{g}_{U,i} \\ g_{U,i}|_{\partial B(x_i^U, \alpha)} &= f|_{\partial B(x_i^U, \alpha)}. \end{aligned}$$

We put these maps together and extend them using f to obtain a map $g \in C(M)$ such that

$$\begin{aligned} g|_{B_{U,V}} &= g_{U,V}, \\ g|_{B(x_i^U, \alpha)} &= g_{U,i}, \\ g|_{M \setminus C} &= f|_{M \setminus C}, \end{aligned}$$

where

$$C := \bigcup \{B_{U,V} \cup B(x_i^U, \alpha) \mid U \in \mathcal{S}_n,\ V \in \mathcal{S}_{n,U},\ i \in \{1, \ldots, l_U\}\}.$$

Note that g may only differ from f on $B_{U,V}$ and $B(x_i^U, \alpha)$, and the images of these sets by both maps are simultaneously contained in $D_{U,V}$ and in $B(f(x_i^U, \frac{\delta}{2})$, respectively. Since the diameters of these sets do not exceed δ, it follows that

$$\rho_0(f, g) \leq \delta < \varepsilon.$$

Moreover, if $f(\overline{U}) \cap \overline{V} \neq \emptyset$ for some $U, V \in \mathcal{S}_n$ then we defined $x_{U,V}$ and $y_{U,V}$ such that $g(x_{U,V}) = y_{U,V}$, where $x_{U,V} \in B_{U,V} \subset U$ and $x_{U,V} \in V$, which implies $g(\overline{U}) \cap \overline{V} \neq \emptyset$. On the other hand, if $f(\overline{U}) \cap \overline{V} = \emptyset$ then also $g(\overline{U}) \cap \overline{V} = \emptyset$, because $\rho_0(f, g) \leq \delta < \gamma$ and $d(f(\overline{U}), \overline{V}) \geq \gamma$.

Finally, for each $U \in S_n$ and $V \in S_{n,U}$, there exists a ball $\tilde{K}_{U,V}$ centered at $x_{U,V}$ small enough to ensure that $\tilde{K}_{U,V} \subset B_{U,V}$ and $g(\tilde{K}_{U,V}) \subset V$ (thanks to the openness of both U and V as well as continuity of g). By moving the center of each of these balls slightly when necessary and decreasing the radius, the family $\{\tilde{K}_{U,V}\}$ can be transformed to another family $\mathcal{K} = \{K_{U,V}\}$ of closed balls, each contained in the corresponding $\tilde{K}_{U,V}$, such that $g(K_{U,V}) \subset D_{U,V}$. The balls $K_{U,V}$ are mutually disjoint, because so are the balls $B_{U,V}$ containing them. Recall that $B_{U,V} \subset U$, and thus $K_{U,V} \subset U$. Moreover, $g(K_{U,V}) \subset g(\tilde{K}_{U,V}) \subset V$. Finally, by (1), (2), and the fact that the map (3) is a surjection,

$$f(\overline{U}) \subset g(\overline{U} \setminus B_{U,V}) \subset g(\overline{U} \setminus K_{U,V}).$$

Since the sets $g(K_{U,V_1}) \subset V_1$ and $g(K_{U,V_2}) \subset V_2$ are disjoint whenever $V_1 \neq V_2$, we also have

$$f(\overline{U}) \subset g(\overline{U} \setminus \bigcup\{K_{U,V} \mid V \in S_{n,U}\}),$$

and thus

$$f(M) = f(\bigcup\{\overline{U} \mid U \in \mathcal{S}_n\}) \subset$$
$$\subset g(\bigcup\{\overline{U} \setminus \bigcup\{K_{U,V} \mid V \in S_{n,U}\} \mid U \in \mathcal{S}_n\}) = g(M \setminus \bigcup \mathcal{K}),$$

which completes the proof. □

Now we are ready to prove density of $\hat{\mathcal{B}}_n$. We keep this proof as elementary as possible.

Proposition 3.10. *The set $\hat{\mathcal{B}}_n$ is dense in $C(M)$. Moreover, the set $\hat{\mathcal{B}}_n \cap S(M)$ is dense in $S(M)$.*

Proof. Let $f \in C(M)$. Let $\varepsilon > 0$. Take $m > n$ such that $\frac{1}{m} < \frac{\varepsilon}{2}$. We are going to modify f on some subsets of M so as to obtain a map $\tilde{f}$ that belongs to $\hat{\mathcal{A}}_m$ and satisfies $\rho_0(f, \tilde{f}) < \varepsilon$.

Apply Lemma 3.9 to f, m, and $\frac{\varepsilon}{2}$ to obtain a map $g \in C(M)$ and a family of closed balls $\mathcal{K} = \{K_{U,V} \mid U \in \mathcal{S}_m,\ V \in \mathcal{S}_{m,U}\}$, where

$$\mathcal{S}_{m,U} = \{V \in \mathcal{S}_m \mid f(\overline{U}) \cap \overline{V} \neq \emptyset\} = \{V \in \mathcal{S}_m \mid g(\overline{U}) \cap \overline{V} \neq \emptyset\}.$$

Then any map $\tilde{f}$ which coincides with g on the set $M \setminus \bigcup \mathcal{K}$ satisfies

$$f(M) \subset g(M \setminus \bigcup \mathcal{K}) = \tilde{f}(M \setminus \bigcup \mathcal{K}).$$

In particular, if $f \in S(M)$ then $\tilde{f} \in S(M)$, provided that $\tilde{f} \in C(M)$.

For each $U \in \mathcal{S}_m$ and $V \in \mathcal{S}_{m,U}$, take a finite family of arbitrary pairwise disjoint closed balls $\{B^W_{U,V} \mid W \in \mathcal{S}_{m,V}\}$, all contained in the interior of $K_{U,V}$. For each $B^W_{U,V}$, take an arbitrary homeomorphism

$$\varphi^W_{U,V} \colon B^W_{U,V} \to \tilde{V},$$

where $\tilde{V}$ is a closed subset contained in V (with V perceived as a ball in $\mathbb{R}^k$) such that $\tilde{V}$ is homeomorphic with $\overline{\mathbb{I}^k}$ (and thus with $B^W_{U,V}$), and $\bigcup\{K_{V,W} | W \in \mathcal{S}_{m,V}\} \subset \operatorname{int} \tilde{V}$. In this way, on each set

$$D_{U,V} := \bigcup\{B^W_{U,V} \mid W \in \mathcal{S}_{m,V}\}$$

we have set up a continuous map

$$\varphi_{U,V} \colon D_{U,V} \to \tilde{V} \subset V,$$

which coincides with $\varphi^W_{U,V}$ on each $B^W_{U,V}$. Note that $D_{U,V}$ is a closed set contained in the interior of $K_{U,V} \subset U$ and V is an open ball (i.e, a convex set in $\mathbb{R}^k$ up to a homeomorphism), hence by Lemma 3.7 we can extend each continuous function

$$g_{U,V} \colon D_{U,V} \cup \partial K_{U,V} \to V,$$

defined by

$$\begin{aligned} g_{U,V}|_{D_{U,V}} &:= \varphi_{U,V}, \\ g_{U,V}|_{\partial K_{U,V}} &:= g|_{\partial K_{U,V}}, \end{aligned}$$

to a continuous function

$$\tilde{f}_{U,V}\colon K_{U,V} \to V$$

that coincides with $g_{U,V}$ on the set $D_{U,V} \cup \partial K_{U,V}$. By putting all the functions $\tilde{f}_{U,V}$ together and extending them on M using g, we obtain a function $\tilde{f} \in C(M)$, such that

$$\tilde{f}|_{K_{U,V}} = \tilde{f}_{U,V}$$

for each $U \in \mathcal{S}_m$, $V \in \mathcal{S}_{m,U}$, and

$$\tilde{f}|_{M\setminus\bigcup\mathcal{K}} = g|_{M\setminus\bigcup\mathcal{K}}.$$

Observe that

$$\rho_0(\tilde{f}, g) \le \max\{\rho_0(\tilde{f}_{U,V}, g|_{K_{U,V}}) \mid U \in \mathcal{S}_m,\ V \in \mathcal{S}_{m,U}\} \le \operatorname{diam} V \le \tfrac{1}{m} < \tfrac{\varepsilon}{2},$$

and thus

$$\rho_0(f, \tilde{f}) \le \rho_0(f, g) + \rho_0(g, \tilde{f}) < \varepsilon.$$

Additionally, notice that

$$f(\overline{U}) \cap \overline{V} \ne \emptyset \iff g(\overline{U}) \cap \overline{V} \ne \emptyset \iff \tilde{f}(\overline{U}) \cap \overline{V} \ne \emptyset$$

where the first equivalence comes from the conclusion of Lemma 3.9, and the second one follows from the fact that $\tilde{f}$ differs from g only on closed balls contained in U whose images by both maps are contained in V for all $U \in \mathcal{S}_m$ and $V \in \mathcal{S}_{m,U}$.

It remains to show that $\tilde{f} \in \hat{\mathcal{A}}_m$. Define the family of pairwise disjoint closed balls

$$B_m := \{B^W_{U,V} \mid U \in \mathcal{S}_m,\ V \in \mathcal{S}_{m,U},\ W \in \mathcal{S}_{m,V}\}.$$

By construction of the balls $B^W_{U,V}$, the family B_m is obviously inscribed into $\mathcal{S}_m$.

For each $U \in \mathcal{S}_m$ and $V \in \mathcal{S}_{m,U}$, take any $W = W_{U,V} \in \mathcal{S}_{m,V}$. Note that

$$\operatorname{int} \tilde{f}(B^W_{U,V}) = \operatorname{int} \tilde{V} \supset \bigcup\{B \in B_m \mid B \subset V\} = \bigcup B_{m,V} =: D.$$

Therefore by Theorem 3.8 applied to $A := K_{U,V}$, $\tilde{f}$, $B := B^W_{U,V}$ and D defined above (note that, as usual, U and V can be considered as subsets of $\mathbb{R}^k$ up to a homeomorphism), there exists a constant $\delta_{U,V} > 0$ such that

$$D \subset h(B^W_{U,V})$$

for all $h \in C(M)$ with $\rho_0(\tilde{f}, h) < \delta_{U,V}$. Take

$$\delta := \min\{\delta_{U,V} \mid U \in \mathcal{S}_m,\ V \in \mathcal{S}_{m,U}\}.$$

Then

$$\bigcup B_{m,V} \subset \bigcap\{h(B) \mid h \in C(M), \rho_0(\tilde{f}, h) < \delta\}$$

for every $U, V \in \mathcal{S}_m$ with $\tilde{f}(\overline{U}) \cap \overline{V} \ne \emptyset$ and for each $B := B^W_{U,V} \in B_{m,U}$ with the chosen $W = W_{U,V}$. Hence $\tilde{f}$ is δ-compatible with B_m and $\mathcal{S}_m$. Moreover, by the construction of $\tilde{f}$, the map $\tilde{f}|_B\colon B \to \tilde{f}(B)$ is a homeomorphism for each $B \in B_m$. This completes the proof. □

As an immediate consequence of Observation 3.3 and Proposition 3.10, we have the following result.

Corollary 3.11. *The sets $\mathcal{B}_n$ and $\bar{\mathcal{B}}_n$ are dense in $C(M)$. Moreover, the sets $\mathcal{B}_n \cap S(M)$ and $\bar{\mathcal{B}}_n \cap S(M)$ are dense in $S(M)$.*

3.4. **Shadowing.** We first show that each map $f \in \mathcal{A}_n$ satisfies the shadowing property with $\varepsilon = \frac{1}{n}$. Then we use this fact to prove the genericity of shadowing.

Proposition 3.12. *Let $n \geq 1$ and $f \in \mathcal{A}_n$. Then there exists $\delta > 0$ such that each δ-pseudo-orbit of f is $\frac{1}{n}$-traced by some orbit of f.*

Proof. Let $\delta > 0$ be a constant such that

$$\delta < \operatorname{dist}(f(\overline{U}), \overline{V})$$

for every $U, V \in S_n$ such that $f(\overline{U}) \cap \overline{V} = \emptyset$.

Let $\{y_i\}_{i\in\mathbb{N}}$ be an arbitrary δ-pseudo-orbit for f. For each $i \in \mathbb{N}$, let U_i be a set in S_n containing the point y_i in its closure. Then we have

$$\operatorname{dist}(f(\overline{U_i}), \overline{U_{i+1}}) \leq \delta,$$

and hence

$$f(\overline{U_i}) \cap \overline{U_{i+1}} \neq \emptyset.$$

By the fact that f is $\hat{\delta}$-compatible with some B_n and $\mathcal{S}_n$ for some $\hat{\delta} > 0$, there exists a sequence of closed balls $\{B_i\}_{i\in\mathbb{N}}$ satisfying

$$B_i \subset U_i \text{ and } f(B_i) \supset B_{i+1} \text{ for all } i \in \mathbb{N}.$$

It is easy to see that then there exists a sequence of points $\{x_i\}_{i\in\mathbb{N}}$ such that

$$f^j(x_i) \in B_j \text{ for } j = 0, 1, \ldots, i.$$

Let x be an accumulation point of $\{x_i\}_{i\in\mathbb{N}}$. Obviously,

$$x \in \bigcap_{i\in\mathbb{N}} f^{-i}(B_i). \tag{4}$$

Hence

$$d(f^i(x), y_i) \leq \operatorname{diam} U_i < \tfrac{1}{n}, \tag{5}$$

which completes the proof. □

Proof of Theorem 1.1. Let

$$\mathcal{B} := \bigcap_{n\in\mathbb{N}} \mathcal{B}_n.$$

By Observation 3.6 and Corollary 3.11, $\mathcal{B}$ is a residual subset of $C(M)$, and also $\mathcal{B} \cap S(M)$ is a residual subset of $S(M)$. It remains to prove that each $f \in \mathcal{B}$ has the shadowing property.

Let $f \in \mathcal{B}$ and $\varepsilon > 0$. Take $n \in \mathbb{N}$ satisfying $\frac{1}{n} < \varepsilon$. Since $f \in \mathcal{B}_n$ by definition of $\mathcal{B}$, $f \in \mathcal{A}_m$ for some $m \geq n$. Then by Proposition 3.12, there exists $\delta > 0$ such that each δ-pseudo-orbit of f is $\frac{1}{m}$-traced by some orbit of f. The fact that $\frac{1}{m} < \varepsilon$ completes the proof. □

3.5. **Periodic shadowing.** Like in Section 3.4, we first prove that each map $f \in \bar{\mathcal{A}}_n$ satisfies the periodic shadowing property with $\varepsilon = \frac{1}{n}$, and then we use this fact to prove the genericity of periodic shadowing.

Proposition 3.13. *Let $n \geq 1$ and $f \in \bar{\mathcal{A}}_n$. Then there exists $\delta > 0$ such that each periodic δ-pseudo-orbit of f is $\frac{1}{n}$-traced by some periodic orbit of f.*

Proof. Let $\delta > 0$ be a constant such that

$$\delta < \operatorname{dist}(f(\overline{U}), \overline{V})$$

for every $U, V \in S_n$ such that $f(\overline{U}) \cap \overline{V} = \emptyset$. Let $\{y_i\}_{i\in\mathbb{N}}$ be a periodic δ-pseudo-orbit for f, with $y_{i+N} = y_i$ for some $N > 0$. For each $i \in \{0, \dots, N-1\}$, let U_i be a set in S_n containing the point y_i in its closure. Put $U_{i+N} := U_i$ for all $i \in \mathbb{N}$ to extend this finite sentence by periodic repetition. Then

$$\operatorname{dist}(f(\overline{U_i}), \overline{U_{i+1}}) \leq \delta,$$

and hence

$$f(\overline{U_i}) \cap \overline{U_{i+1}} \neq \emptyset.$$

By the fact that f is $\hat{\delta}$-compatible with some B_n and $\mathcal{S}_n$ for some $\hat{\delta} > 0$, there exists a sequence of closed balls $\{B_i\}_{i\in\mathbb{N}}$ satisfying

$$B_i = B_{i+N},\ B_i \subset U_i,\ \text{and } f(B_i) \supset B_{i+1} \text{ for all } i \in \mathbb{N}.$$

By the strong δ-compatibility of f, the balls B_i can be chosen in such a way that

$$B_0 \overset{f,w_1}{\Longrightarrow} B_1 \overset{f,w_2}{\Longrightarrow} \dots \overset{f,w_N}{\Longrightarrow} B_N = B_0,$$

for some nonzero integers $w_1, \dots, w_N$. Hence, by Theorem 2.10, there exists a periodic point $x \in \operatorname{int} B_0$ such that

$$f^i(x) \in \operatorname{int} B_i \text{ for } i \in \{1, \dots, N\}.$$

Obviously,

$$d(f^i(x), y_i) \leq \operatorname{diam} U_i < \tfrac{1}{n},$$

which completes the proof. □

Proof of Theorem 1.2. The proof is essentially the same as proof of Theorem 1.1, except that we use the sets $\bar{\mathcal{A}}_n$ instead of $\mathcal{A}_n$, and the sets $\bar{\mathcal{B}}_n$ instead of $\mathcal{B}_n$. The corresponding set $\bar{\mathcal{B}} := \bigcap_{n\in\mathbb{N}} \bar{\mathcal{B}}_n$ is residual by Observation 3.6 and Corollary 3.11, and one can prove that each $f \in \bar{\mathcal{B}}$ has the periodic shadowing property using Proposition 3.13. □

3.6. **Inverse shadowing.** Analogously to the previous two sections, we first prove that each map in $\mathcal{A}_n$ satisfies the inverse shadowing property with respect to the family $\mathcal{T}_S$ with $\varepsilon = \frac{1}{n}$, and then we use this fact to prove that the inverse shadowing property with respect to this class is generic.

Proposition 3.14. *Let $n \geq 1$ and $f \in \mathcal{A}_n$. Then there exists $\delta > 0$ such that for every $x \in M$ and for every $\chi \in \mathcal{T}_{S,\delta}$ there exists $y \in M$ satisfying*

$$d(f^i(x), \chi(y)_i) < \tfrac{1}{n} \text{ for every } i \in \mathbb{N}.$$

Proof. The proof is similar to the proof of Proposition 3.12, hence we will skip some details.

Let $\delta > 0$ be a constant such that each $f \in \mathcal{A}_n$ is δ-compatible with some B_n and with $\mathcal{S}_n$. We can additionally assume that

$$\delta < \operatorname{dist}(f(\overline{U}), \overline{V})$$

for every $U, V \in S_n$ such that $f(\overline{U}) \cap \overline{V} = \emptyset$.

Let $\chi \in \mathcal{T}_{S,\delta}$ be an arbitrary δ-method. Let $x \in M$, and for each $i \in \mathbb{N}$, let $U_i \in S_n$ be a set containing $f^i(x)$ in its closure. Let $\{\psi_i\}_{i\in\mathbb{N}}$ be the sequence of continuous maps that defines χ.

From the definition of $\mathcal{T}_{S,\delta}$, it follows that $\rho_0(f, \psi_i) < \delta$ for all $i \in \mathbb{N}$. The fact that f is δ-compatible with B_n and $\mathcal{S}_n$ implies that there exists a sequence of closed balls $\{B_i\}_{i\in\mathbb{N}}$ satisfying

$$B_i \subset U_i \text{ and } \psi_i(B_i) \supset B_{i+1} \text{ for all } i \in \mathbb{N}.$$

Then there exists

$$y \in \bigcap_{i\in\mathbb{N}} (\psi_{i-1} \circ \cdots \circ \psi_0)^{-i}(B_i),$$

and hence

$$d(f^i(x), \chi(y)_i) \leq \operatorname{diam} U_i < \tfrac{1}{n},$$

which completes the proof. □

Observation 3.15. *Since $\mathcal{T}_{H,\delta} \subset \mathcal{T}_{S,\delta}$ for all $\delta > 0$ (see Observation 2.7), the same result as Proposition 3.14 also holds true for the family $\mathcal{T}_H$.*

Proof of Theorem 1.3. The proof is the same as the proof of Theorem 1.1, except that we use Proposition 3.14 and Observation 3.15 instead of Proposition 3.12. Note that, in particular, we use the same sets $\mathcal{A}_n$ and $\mathcal{B}_n$ as those that appear in the proof of Theorem 1.1. □

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