Excision-preserving cubical approach to the algorithmic computation of the discrete Conley index

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Abstract

We introduce a new approach to the algorithmic computation of the Conley index for continuous maps. We use the technique of splitting an index pair into two layers which is inspired by the work of Mrozek, Reineck & Srzednicki (Trans. AMS 352 (2000) 4171–4194). The main advantage of our construction over the approach based directly on the one introduced by Mischaikow, Mrozek & Pilarczyk (Foundations Comp. Math. 5 (2005) 199–229) is that our cubical sets have the excision property. Moreover, our solution has some advantages in comparison to the approach recently proposed by Mrozek (Foundations Comp. Math. 6 (2006) 457–493).[1]

1 Introduction

The Conley index [4, 7, 12, 24] is a topological tool for the study of isolated invariant sets in continuous and discrete dynamical systems. The definition of the Conley index is based upon the notion of an index pair and will be explained later in this section. Introducing a cubical grid in $\mathbb{R}^n$ and enclosing a continuous map in a combinatorial cubical multivalued map (also explained later) allows one to compute index pairs automatically [18, 25]. Cubical homology [8, 10, 20] can be further used to effectively compute the homological version of the index as defined in [12]. In this way, the Conley index can be used in computer-assisted analysis of qualitative behavior of dynamical systems ([3, 5, 11, 16, 17], to mention a few examples).

Unfortunately, sometimes the combinatorial objects obtained in the algorithmic construction of an index pair are not suitable for direct computation of the homological Conley index. Due to the overestimates in the combinatorial

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map, the inclusion map that appears in the definition of the index map is not an excision in some cases, and thus the index map is not properly defined. (These maps are defined further; see formulas (2) and (4) in this section.) The example depicted in Figure 1 illustrates such a situation.

One way to overcome this difficulty is to impose more restrictive conditions on a combinatorial index pair, as proposed in [11], or to decrease the index pair in size, as discussed at the end of this section. Both solutions, however, either limit the applicability of the computational approach to the Conley index, or lead to complications in the algorithmic homology computation (because of the necessity to deal with more general cubical sets), which we would like to avoid.

In this paper we introduce a new alternative method for treating the constructed index pair and index map, which allows for using them to compute the homological Conley index, even if the inclusion in question is not an excision. It seems to be easier and more efficient to deal with computationally, and even allows one to use the already existing algorithms [10] and software [20] for this purpose, with only minor modifications either to the algorithms, or to the processed data, both implemented for instant use and discussed in Section 5.

1.1 Index pairs and the Conley index

Let $\mathbb{Z}$ and $\mathbb{R}$ denote the sets of integers and real numbers, respectively. Although the following definitions can be stated for an arbitrary locally compact metric space, we restrict our attention to $\mathbb{R}^n$ in order to avoid unnecessary complications. For a set $A \subset \mathbb{R}^n$ we denote its closure and its interior by $\operatorname{cl} A$ and $\operatorname{int} A$, respectively.

Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map. Although it is enough to assume that f is only defined on some subset of $\mathbb{R}^n$, for simplicity of notation we will not discuss this general case.

Given a compact set $N \subset \mathbb{R}^n$, the *invariant part* of N is defined as

$$\begin{aligned}\operatorname{Inv} N := \{x \in N \ :\ &\text{there exists a sequence } \{x_n\}_{n\in\mathbb{Z}} \text{ in } N \\ &\text{such that } x_0 = x \text{ and } x_{n+1} = f(x_n) \text{ for all } n \in \mathbb{Z}\}.\end{aligned}$$

The set N is called an *isolating neighborhood* if $\operatorname{Inv} N \subset \operatorname{int} N$.

A pair $P := (P_1, P_2)$ of compact subsets of $\mathbb{R}^n$ is called a *topological pair* in $\mathbb{R}^n$ if $P_2 \subset P_1$. If $Q := (Q_1, Q_2)$ is also a topological pair, then by $f\colon P \to Q$ we denote such a map $f\colon P_1 \to Q_1$ that $f(P_2) \subset Q_2$. A continuous map $h\colon P \to Q$ is said to be a *homeomorphism* between P and Q if $h\colon P_1 \to Q_1$ is a homeomorphism and $h(P_2) = Q_2$.

Definition 1.1 (see [24]) *A topological pair (P_1, P_2) in $\mathbb{R}^n$ is called an* index pair *(with respect to f) if the following conditions are satisfied:*

- *(a)* $f(P_1 \setminus P_2) \subset P_1$,
- *(b)* $f(P_2) \cap P_1 \subset P_2$,
- *(c)* $\operatorname{Inv}\big(\operatorname{cl}(P_1 \setminus P_2)\big) \subset \operatorname{int}(P_1 \setminus P_2)$.

If (P_1, P_2) is an index pair with respect to f, then $P_1 \cup f(P_1) = P_1 \cup f(P_2)$, and the map

$$f_P\colon (P_1, P_2) \to (P_1 \cup f(P_2), P_2 \cup f(P_2)), \tag{1}$$

which sends a point x to $f(x)$, is well defined (and obviously continuous). Moreover,

$$\big(P_1 \cup f(P_2)\big) \setminus P_1 = \big(P_2 \cup f(P_2)\big) \setminus P_2,$$

and therefore the quadruple $\big(P_1, P_2, P_1 \cup f(P_2), P_2 \cup f(P_2)\big)$ has the excision property in the following sense.

Definition 1.2 *Let (P_1, P_2) and (Q_1, Q_2) be topological pairs such that $P_1 \subset Q_1$ and $P_2 \subset Q_2$. We say that the quadruple (P_1, P_2, Q_1, Q_2) has the excision property if $Q_1 \setminus P_1 = Q_2 \setminus P_2$.*

Intuitively, the excision property says that (Q_1, Q_2) is an extension of (P_1, P_2) obtained by expanding P_2 in such a way that $P_1 \setminus P_2$ is not touched. This property is crucial for the definition of the Conley index based on the index pair (P_1, P_2) and the map f_P defined by (1).

The Conley index [4] originally defined for flows was transferred to the case of maps by Mrozek [12] and generalized by Szymczak [24]. Since these generalizations are quite complicated, either using Alexander-Spanier cohomology or formulated using the abstract category theory, an apparently more accessible definition of the Conley index for maps was introduced by Franks and Richeson [7] which is claimed to be equivalent to the one posed by Szymczak (see [7, Proposition 8.1 and 8.2]). However, although the notion of shift equivalence used in [7] appeals to intuition, it is also expressed in the abstract category language similar to [24], and thus turns out to be not much easier than the latter.

Our paper is aimed at developing a construction useful for an actual computational method, and therefore in the algorithmic computations and examples we chose to use Mrozek's construction and the (cubical) homology functor, because this seems to be the only approach which can be dealt with algorithmically. Indeed, efficient homology computation algorithms for maps are known in this context (see [8, 10]), and their implementation is freely available (see [20]), whereas computing a canonical representation of a shift equivalence class seems to be an open problem [K. Mischaikow, personal communication]. In the theoretical justification of the correctness of our method we are going to use the Alexander-Spanier cohomology functor (see [21]) in order to make our reasoning precise and mathematically sound. The Reader not familiar with Alexander-Spanier cohomology theory can instantly skip these parts without loss of the core idea of the paper.

The definition of the (co)homological Conley index is based on the index pair (P_1, P_2), the map f_P defined by (1), and the inclusion map

$$i_P \colon (P_1, P_2) \to \big(P_1 \cup f(P_2), P_2 \cup f(P_2)\big). \tag{2}$$

We will now recall the definition of the cohomological Conley index (the homological Conley index is defined in a similar way).

Let H^* denote the Alexander-Spanier cohomology functor. We denote homomorphisms induced in cohomology by continuous maps by appending a superscript asterisk to the map symbol; in particular, the map f_P induces the following homomorphism in cohomology:

$$f_P^* \colon H^*\big(P_1 \cup f(P_2), P_2 \cup f(P_2)\big) \to H^*(P_1, P_2). \tag{3}$$

Since the quadruple $\big(P_1, P_2, P_1 \cup f(P_2), P_2 \cup f(P_2)\big)$ has the excision property as formulated in Definition 1.2, the inclusion i_P is an excision for the Alexander-Spanier cohomology (see [21], Theorem 6.6.5), and therefore it induces an isomorphism i_P^* in cohomology. We would like to remark that i_P may not be an excision if other (co)homology theories are considered (e.g., singular homology); therefore, using Alexander-Spanier cohomology in the definition of the Conley index is justified by its strong excision property.

Since the homomorphism i_P^* is invertible, the map

$$I_P^* := f_P^* \circ (i_P^*)^{-1} \colon H^*(P_1, P_2) \to H^*(P_1, P_2) \tag{4}$$

is well defined; it is called the *index map* (cf. [12]).

The *cohomological Conley index* of (P_1, P_2) is defined as the Leray reduction of $\big(H^*(P_1, P_2), I_P^*\big)$ (see [24] or [12] for details). It does not depend on the choice of an index pair, but only on the isolated invariant set $\operatorname{Inv}\big(\operatorname{cl}(P_1 \setminus P_2)\big)$; that is to say, if for another index pair (Q_1, Q_2) we have $\operatorname{Inv}\big(\operatorname{cl}(P_1 \setminus P_2)\big) = \operatorname{Inv}\big(\operatorname{cl}(Q_1 \setminus Q_2)\big)$, then the Conley indices of (P_1, P_2) and of (Q_1, Q_2) are the same.

1.2 Combinatorial approach to the Conley index

Although our reasoning is valid for a general class of acyclic grids on locally compact metric spaces, for clarity of presentation we restrict our attention to cubical sets (defined below) based on a rectangular grid in $\mathbb{R}^n$. As it will be seen, the cubical sets we deal with are compact polyhedra, so all the (co)homology theories are equivalent for them, and therefore we can use the simpler cubical homology instead of Alexander-Spanier cohomology in the illustrations and actual computations without loss of generality if we consider these sets and continuous maps between them.

We cover the entire phase space $\mathbb{R}^n$ with a uniform grid of cubes. Since rescaling does not change the quantities we compute, for simplicity we assume that the cubes are of unit size:

$$\mathcal{K} := \Big\{ \prod_{i=1}^{n} [l_i, l_i + 1] : l_i \in \mathbb{Z} \Big\}$$

Every finite subset $\mathcal{A}$ of $\mathcal{K}$ represents a compact subset of $\mathbb{R}^n$:

$$|\mathcal{A}| := \bigcup_{Q \in \mathcal{A}} Q$$

If $A \subset \mathbb{R}^n$ is such that $A = |\mathcal{A}|$ for some finite $\mathcal{A} \subset \mathcal{K}$, then A is called a *(full) cubical set*.

Given two sets X, Y, by $F \colon X \multimap Y$ we denote a *multivalued map*, i.e., a map $F \colon X \to 2^Y$ such that $F(x) \subset Y$ for each $x \in X$. If $\mathcal{X}, \mathcal{Y} \subset \mathcal{K}$ are finite then a multivalued map $\mathcal{F} \colon \mathcal{X} \multimap \mathcal{Y}$ is called a *combinatorial cubical multivalued map* (or a *combinatorial map* for short). We say that a combinatorial map $\mathcal{F} \colon \mathcal{X} \multimap \mathcal{Y}$ is a *combinatorial representation* of a continuous map $f \colon |\mathcal{X}| \to |\mathcal{Y}|$ if

$$f(Q) \subset \operatorname{int} \big|\mathcal{F}(Q)\big| \tag{5}$$

for all $Q \in \mathcal{X}$ (cf. [25]). If a combinatorial representation $\mathcal{F}\colon (\mathcal{X}, \mathcal{A}) \multimap (\mathcal{Y}, \mathcal{B})$ of $f\colon (|\mathcal{X}|, |\mathcal{A}|) \to (|\mathcal{Y}|, |\mathcal{B}|)$ is acyclic (see [10] for details), then it can be used to compute automatically (i.e., on the computer) the homomorphism f_* induced by f in homology (see [10, 20]).

Definition 1.3 (see [17]) *We say that a pair* $(\mathcal{P}_1, \mathcal{P}_2)$ *of finite subsets of* $\mathcal{X}$ *such that* $\mathcal{P}_2 \subset \mathcal{P}_1$ *is a* combinatorial index pair *with respect to a combinatorial map* $\mathcal{F}\colon \mathcal{X} \multimap \mathcal{Y}$ *if the following conditions hold:*

- *(a)* $\mathcal{F}(\mathcal{P}_1 \setminus \mathcal{P}_2) \subset \mathcal{P}_1$,
- *(b)* $\mathcal{F}(\mathcal{P}_2) \cap \mathcal{P}_1 \subset \mathcal{P}_2$.

Proposition 1.4 (see [17]) *If* $(\mathcal{P}_1, \mathcal{P}_2)$ *is an index pair with respect to a combinatorial representation* $\mathcal{F}$ *of* f*, then* $|\mathcal{P}| := \big(|\mathcal{P}_1|, |\mathcal{P}_2|\big)$ *is an index pair with respect to* f*.*

If the quadruple $\big(|\mathcal{P}_1|, |\mathcal{P}_2|, |\mathcal{P}_1 \cup \mathcal{F}(\mathcal{P}_2)|, |\mathcal{P}_2 \cup \mathcal{F}(\mathcal{P}_2)|\big)$ has the excision property (as in Definition 1.2), then the combinatorial representation $\mathcal{F}$ of f can be used to compute the Conley index of $(P_1, P_2) := (|\mathcal{P}_1|, |\mathcal{P}_2|)$. Namely, one must introduce combinatorial analogues of the maps f_P and i_P, defined by (3) and (2), respectively, based on the combinatorial representation $\mathcal{F}$ of f, defined as follows. The map f_P is replaced by

$$f_{\mathcal{P},\mathcal{F}}\colon \big(|\mathcal{P}_1|, |\mathcal{P}_2|\big) \to \big(|\mathcal{P}_1 \cup \mathcal{F}(\mathcal{P}_2)|, |\mathcal{P}_2 \cup \mathcal{F}(\mathcal{P}_2)|\big)$$

given by exactly the same formula as $f_{|\mathcal{P}|}$, but note that with a different codomain. Similarly, in place of i_P one must introduce the inclusion

$$i_{\mathcal{P},\mathcal{F}}\colon \big(|\mathcal{P}_1|, |\mathcal{P}_2|\big) \to \big(|\mathcal{P}_1 \cup \mathcal{F}(\mathcal{P}_2)|, |\mathcal{P}_2 \cup \mathcal{F}(\mathcal{P}_2)|\big) \tag{6}$$

which again differes from $i_{|\mathcal{P}|}$ by its codomain only.

Unfortunately, it turns out that in general, for a combinatorial index pair, it may sometimes happen that

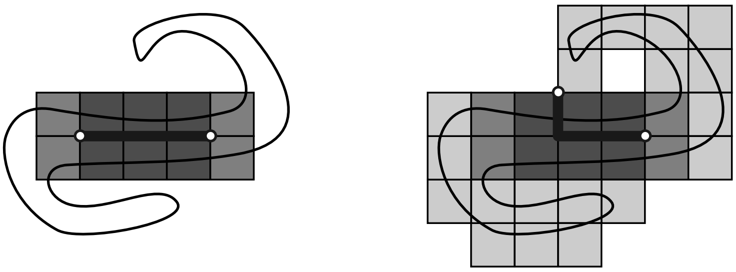
$$|\mathcal{P}_1 \cup \mathcal{F}(\mathcal{P}_2)| \setminus |\mathcal{P}_1| \neq |\mathcal{P}_2 \cup \mathcal{F}(\mathcal{P}_2)| \setminus |\mathcal{P}_2|$$

and, as a consequence, $i_{\mathcal{P},\mathcal{F}}$ may not induce an isomorphism in homology, making the would-be index map $I_{\mathcal{P},\mathcal{F}*} := (i_{\mathcal{P},\mathcal{F}*})^{-1} \circ f_{\mathcal{P},\mathcal{F}*}$ (where the subscript asterisk indicates the corresponding map in homology) improperly defined. An example of such a situation is illustrated in Figure 1. If this happens then it is not obvious how to obtain an appropriate index map based on the combinatorial objects $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{F}$.

Note that the example shown in Figure 1 is especially misleading, because

$$H_*\big(|\mathcal{P}_1|, |\mathcal{P}_2|\big) \simeq H_*\big(|\mathcal{P}_1 \cup \mathcal{F}(\mathcal{P}_2)|, |\mathcal{P}_2 \cup \mathcal{F}(\mathcal{P}_2)|\big),$$

where H_* denotes the (cubical) homology functor (see [8]). This isomorphism, however, is not induced by the inclusion. Namely, the generator of $H_1\big(|\mathcal{P}_1|, |\mathcal{P}_2|\big)$ indicated in picture (a) is mapped by $i_{\mathcal{P},\mathcal{F}*}$ to zero, and another generator which surrounds the hole in picture (b) appears. In general, there is no reason why the spaces $H_*\big(|\mathcal{P}_1|, |\mathcal{P}_2|\big)$ and $H_*\big(|\mathcal{P}_1 \cup \mathcal{F}(\mathcal{P}_2)|, |\mathcal{P}_2 \cup \mathcal{F}(\mathcal{P}_2)|\big)$ should be isomorphic, and often they are not.



(a) (b)

Figure 1: Example index pair in which i_P defined by (6) does not induce an isomorphism in homology (the thick lines with white endpoints represent generators of the first homology group). (a) $\mathcal{P}_1 \setminus \mathcal{P}_2$ consists of the dark grey squares, $\mathcal{P}_2$ is plotted as medium grey squares, and the border of $f(|\mathcal{P}_1|)$ is indicated with the smooth line. (b) Additionally, $\mathcal{F}(\mathcal{P}_1) \setminus \mathcal{P}_1$ is plotted as light grey squares.

It is clear that the problem with the lack of the excision property arises due to an overestimation of $f(|\mathcal{P}_2|)$ by $|\mathcal{F}(\mathcal{P}_2)|$ which may have a nonempty intersection with $|\mathcal{P}_1| \setminus |\mathcal{P}_2|$. A solution to this issue proposed in [11] is to impose more restrictive assumptions on the combinatorial index pair[2] (see Definition 7.2 and Theorem 8.1 in [11]). These assumptions, however, may be more difficult to satisfy (for instance, the example illustrated in Figure 1 is not a combinatorial index pair in the sense of [11]). In order to verify them it may also be necessary to compute $\mathcal{F}$ not only on $\mathcal{P}_1$, but on some larger set $\mathcal{N}$, and this computation may be costly in some cases. A specific example that illustrates this issue is discussed in Section 5.

Moreover, using sets built of full cubes is advantageous because of efficient geometric reduction techniques which have already been developed and their software implementation [20] is available. This is in contrast to the approach based on the idea of weak index pairs introduced in [11] in which a more general class of cubical sets appears. Those sets are built of both full cubes and their faces, which makes some geometric reduction techniques inapplicable. Therefore, we are not interested in that solution either.

In the subsequent sections, we introduce an alternative approach which allows one to use a combinatorial index pair as in Definition 1.3, built of full cubical sets, for the computation of the Conley index. The idea is to lift the set $|\mathcal{P}_1|$ up to a higher level, so that $|\mathcal{F}(\mathcal{P}_2)|$ (which remains at the base level) does not have a chance to intersect it and cause trouble. A formal definition of this operation is based upon considering two-layer sets with the identification of $|\mathcal{P}_2|$ on both layers, and is described in details in Sections 2 and 3, with a formal justification of the correctness of using it to compute the Conley index postponed to Section 4. Note that an idea similar in spirit to this construction was already considered in [13] in the proof of Theorem 3, as we were informed by M. Mrozek after having shown him our manuscript.

[2] In fact, our Definition 1.3 is not a generalization of Definition 7.2 in [11]; it is different.

2 Double-layer Ω-sets and the Ω-map

In this section we introduce a construction of two-layer sets built upon an index pair, and we transfer the map f_P onto these sets. We also prove that the transferred index pair and its image satisfy the excision property. The formal justification of the correctness of this construction for the purpose of the computation of the Conley index is postponed to Section 4. A combinatorial version of this construction applicable to combinatorial index pairs will be introduced in Section 3.

2.1 Definition of Ω-sets

Given three compact sets $R_0, R_1, R_2 \subset X$ such that $R_2 \subset R_0 \cap R_1$, let $\sim$ denote the relation in $R_1 \times 1 \cup R_0 \times 0$ which identifies $(x, 1)$ with $(x, 0)$ for each $x \in R_2$. Inspired by the ideas from [14], we define the following set:

$$\Omega_{R_2}(R_1, R_0) := R_1 \times 1 \cup R_0 \times 0/_{\sim}.$$

In $\Omega_{R_2}(R_1, R_0)$ we introduce the usual quotient topology induced by the projection

$$q\colon R_1 \times 1 \cup R_0 \times 0 \to \Omega_{R_2}(R_1, R_0)$$

which sends a point (x, i) from the domain to its equivalence class $[x, i]$ with respect to the relation $\sim$. We call a set constructed in this way a *(double-layer) Ω-set*. Once it is clear from the context what set R_2 is considered, we shall drop the subscript and write $\Omega(R_1, R_0)$ instead of $\Omega_{R_2}(R_1, R_0)$.

Let $P := (P_1, P_2)$ be an index pair. Fix $R_2 := P_2$. To shorten the notation, we use the following symbols (see Figure 2 for an illustration):

$$\begin{aligned}
S(P_1) &:= P_1 \cup f(P_1) \quad \big(= P_1 \cup f(P_2) \text{ by Def. 1.1 (a)}\big) \\
S(P_2) &:= P_2 \cup f(P_2) \\
S(P) &:= \big(S(P_1), S(P_2)\big) \\
P_\Omega &:= \big(\Omega(P_1, P_2), \Omega(P_2, P_2)\big) \\
S(P_\Omega) &:= \big(\Omega(P_1, S(P_2)), \Omega(P_2, S(P_2))\big)
\end{aligned}$$

Figure 2: The set $\Omega\big(P_1, S(P_2)\big)$ for $P_1 = [2, 8] \subset \mathbb{R}$ and $P_2 = [2, 4] \cup [6, 8]$ with $f(P_2) \setminus P_2 = [0, 2] \cup [8, 10]$: (a) The sets $P_1 \times 1$ (the upper layer), $P_2 \times 0$ (the dark grey part of the lower layer), $(f(P_2) \setminus P_2) \times 0$ (the light grey part of the lower layer), and the identification relation $\sim$ (indicated by arrows); (b) An intuitive illustration of the quotient space $\Omega\big(P_1, S(P_2)\big)$.

2.2 Definition of the Ω-map

As in [22] and [23], let us define the map $\widetilde{f_P}\colon P_\Omega \to S(P_\Omega)$ induced by $f_P\colon P \to S(P)$ in the following way:

$$
\widetilde{f_P}([x,i]) := \begin{cases} [f_P(x),0] & \text{if } x \in P_2, \\ [f_P(x),1] & \text{if } x \in P_1 \setminus P_2. \end{cases} \tag{7}
$$

We call this map the Ω-*map*.

Analogously as in [22], we prove the following result.

Proposition 2.1 *The map $\widetilde{f_P}$ is well defined and continuous.*

Proof: Property (a) in Definition 1.1 implies that $\widetilde{f_P}$ is well defined.

In order to prove the continuity of $\widetilde{f_P}$ we will show that $\widetilde{f_P} \circ q$ is continuous (see [6], Theorem 2.4.2).

First, note that the following holds:

$$
f\big(\operatorname{cl}(P_1 \setminus P_2)\big) \subset \operatorname{cl} f(P_1 \setminus P_2) \subset \operatorname{cl} P_1 \subset P_1, \tag{8}
$$

where the first inclusion is a consequence of the continuity of f, the second one follows from (a) in Definition 1.1, and the third one is trivial.

Let $x \in \operatorname{cl}(P_1 \setminus P_2)$ such that $x \notin P_1 \setminus P_2$. Then $x \in P_2$ and $\widetilde{f_P}([x,i]) = [f_P(x),0]$ by (7). On the other hand, inclusion (8) implies that $f_P(x) \in P_1$, and by property (b) in Definition 1.1, $f_P(x) \in P_2$. Therefore, $(f_P(x),0) \sim (f_P(x),1)$, and, consequently, $[f_P(x),0] = [f_P(x),1]$. As a result, $\widetilde{f_P}([x,i]) = [f_P(x),1]$ for all $x \in \operatorname{cl}(P_1 \setminus P_2)$, not only for $x \in P_1 \setminus P_2$, as defined in (7).

Since the restrictions of the map $\widetilde{f_P} \circ q$ to any of the three closed sets $P_2 \times \{0\}$, $\operatorname{cl}(P_1 \setminus P_2) \times \{1\}$ and $P_2 \times \{1\}$ which cover its domain are continuous, the map $\widetilde{f_P} \circ q$ itself is continuous, too. $\square$

2.3 Correspondence between the two-layer objects and the original ones

In order to establish the relation between the constructed Ω-sets and Ω-map and the original index pair and the map f_P, let us define the following maps

$$
h\colon P_1 \ni x \mapsto [x,1] \in \Omega(P_1,P_2) \tag{9}
$$

$$
S(h)\colon S(P_1) \ni x \mapsto \begin{cases} [x,1] & \text{if } x \in P_1 \\ [x,0] & \text{if } x \in S(P_2) \end{cases} \in \Omega\big(P_1, S(P_2)\big) \tag{10}
$$

Proposition 2.2 *The maps (9) and (10) are homeomorphisms of pairs, $h\colon P \to P_\Omega$ and $S(h)\colon S(P) \to S(P_\Omega)$, respectively.*

Proof: Note that $S(h)$ is well defined, because $[x,0] = [x,1]$ for $x \in P_2$ and there is no such x that would both belong to $f(P_2)$ and $P_1 \setminus P_2$ by Definition 1.1, property (b).

To prove that $S(h)$ is continuous, note that the restrictions of $S(h)$ to both P_1 as well as to $S(P_2)$ are continuous as compositions of continuous maps: the embeddings

$$S(P_2) \hookrightarrow S(P_2) \times 0 \qquad \text{and} \qquad P_1 \hookrightarrow P_1 \times 1$$

and the projection q. Since P_1 and $S(P_2)$ are closed and their union is the entire domain of $S(h)$, this proves the continuity of $S(h)$.

The inverse map

$$S(h)^{-1} \colon \Omega\big(P_1, S(P_2)\big) \to S(P_1),$$

expressed as $S(h)^{-1}([x,i]) = x$ is well defined. It is continuous as a map defined on the quotient space iff

$$S(h)^{-1} \circ q \colon P_1 \times 1 \cup S(P_2) \times 0 \to S(P_1)$$

is continuous (see [6] Theorem 2.4.2), and the latter is obvious.

Last but not least, notice that $S(h)$ is in fact a homeomorphism between topological pairs (as explained in Section 1), because

$$S(h)\big(S(P_2)\big) = \Omega\big(P_2, S(P_2)\big).$$

The proof of the fact that h is a suitable homeomorphism is left to the reader.
□

Lemma 2.3 *Under the already established notation, the following diagram commutes:*

$$\begin{array}{ccc} P & \xrightarrow{f_P} & S(P) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle S(h)} \\ P_\Omega & \xrightarrow[\widetilde{f_P}]{} & S(P_\Omega) \end{array}$$

Proof: Take any $x \in P_1$. We will prove that

$$S(h)\big(f_P(x)\big) = \widetilde{f_P}\big(h(x)\big).$$

Namely, if $x \in P_1 \setminus P_2$, then $f(x) \in P_1$ by property (a) in Definition 1.1, and therefore $S(h)\big(f_P(x)\big) = [f_P(x), 1]$. On the other hand, $h(x) = [x, 1]$, and $\widetilde{f_P}([x,1]) = [f_P(x), 1]$. If $x \in P_2$, then $f(x) \in S(P_2)$, and $S(h)\big(f_P(x)\big) = [f_P(x), 0]$. By (7), in this case $\widetilde{f_P}\big(h(x)\big) = \widetilde{f_P}([x,1]) = [f_P(x), 0]$. This completes the proof. □

Similarly to Lemma 2.3, one can prove the following lemma which will play an important role in Section 4.

Lemma 2.4 *Under the already established notation, the following diagram commutes:*

$$\begin{array}{ccc} S(P) & \xleftarrow{i_P} & P \\ {\scriptstyle S(h)}\downarrow & & \downarrow{\scriptstyle h} \\ S(P_\Omega) & \xleftarrow[\widetilde{i_P}]{} & P_\Omega \end{array}$$

2.4 The excision property

We are now ready to prove that the constructed Ω-sets are equally good as the original index pair for the Conley index.

Theorem 2.5 *The quadruple* $\big(P_\Omega, S(P_\Omega)\big)$ *satisfies the excision property.*

Proof: We will prove that

$$\Omega\big(P_1, S(P_2)\big) \setminus \Omega(P_1, P_2) = \Omega\big(P_2, S(P_2)\big) \setminus \Omega(P_2, P_2). \tag{11}$$

To shorten the notation, denote the set on the left hand side of the equation (11) by L, and the set on the right hand side by R. Consider $[x, i]$ that belongs either to L or to R. Since $\Omega\big(P_2, S(P_2)\big) \subset \Omega\big(P_1, S(P_2)\big)$, we know that $[x, i] \in \Omega\big(P_1, S(P_2)\big)$, and thus

$$x \in P_1 \cup S(P_2) = (P_1 \setminus P_2) \cup P_2 \cup \big(S(P_2) \setminus P_2\big).$$

If $x \in P_1 \setminus P_2$, then $i = 1$. Note that $[x, 1]$ neither belongs to L (because $[x, 1] \in \Omega(P_1, P_2)$), nor to R (because $[x, 1] \notin \Omega\big(P_2, S(P_2)\big)$).

If $x \in P_2$, then $[x, i]$ belongs to $\Omega(P_1, P_2)$, as well as to $\Omega(P_2, P_2)$, so again $[x, i] \notin L$ and $[x, i] \notin R$.

Finally, if $x \in S(P_2) \setminus P_2$, then $i = 0$, and $[x, 0] \in \Omega\big(P_1, S(P_2)\big)$, but $[x, 0] \notin \Omega(P_1, P_2)$ and therefore $[x, 0] \in L$. On the other hand, $[x, 0] \in \Omega\big(P_2, S(P_2)\big)$, but $[x, 0] \notin \Omega(P_2, P_2)$, so $[x, 0] \in R$, which completes the proof. □

In Section 4 we will show that the double-layer version P_Ω of the index pair gives rise to the same Conley index as P.

3 Double-layer combinatorial Ω-sets and the combinatorial Ω-map

In this section we transfer the notion of double-layer Ω-sets and the Ω-map to the combinatorial setting, and we prove that the combinatorial index pair as in Definition 1.3 shifted into the double-layer setting gives rise to a quadruple which satisfies the excision property, regardless of the overestimates which caused a problem in the examples discussed in the Introduction. The justification of the fact that the Conley index computed in the double-layer setting coincides with the Conley index for the original index pair is postponed to Section 4. We begin by transferring many notions related to cubical sets to the double-layer cubical sets; the reason for this is that formally the set (12) defined below is not a cubical set.

3.1 Definition of combinatorial Ω-sets

As in Section 2, given three finite sets $\mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_2 \subset \mathcal{K}$ such that $\mathcal{R}_2 \subset \mathcal{R}_0 \cap \mathcal{R}_1$, we define the set

$$\Omega_{\mathcal{R}_2}(\mathcal{R}_1, \mathcal{R}_0) := \mathcal{R}_1 \times 1 \cup \mathcal{R}_0 \times 0/_\sim, \tag{12}$$

where $(Q, 1) \sim (Q, 0)$ iff $Q \in \mathcal{R}_2$. Sets constructed in this way are called by us *combinatorial* Ω*-sets*. Like in Section 2, we shall write $\Omega(\mathcal{R}_1, \mathcal{R}_0)$ instead of $\Omega_{\mathcal{R}_2}(\mathcal{R}_1, \mathcal{R}_0)$ once $\mathcal{R}_2$ is clear from the context.

Combinatorial Ω-sets represent Ω-sets in the following way:

$$|\Omega_{\mathcal{R}_2}(\mathcal{R}_1,\mathcal{R}_0)| := \Omega_{|\mathcal{R}_2|}(|\mathcal{R}_1|,|\mathcal{R}_0|),$$

and subsets of combinatorial Ω-sets represent subsets of the corresponding Ω-sets; namely, if $\mathcal{Q} \subset \Omega_{\mathcal{R}_2}(\mathcal{R}_1,\mathcal{R}_0)$, then

$$|\mathcal{Q}| := \{[x,i] : x \in Q, [Q,i] \in \mathcal{Q}\} \subset \Omega_{|\mathcal{R}_2|}(|\mathcal{R}_1|,|\mathcal{R}_0|).$$

Let $(\mathcal{P}_1,\mathcal{P}_2)$ be an index pair with respect to a combinatorial representation $\mathcal{F}$ of f. From now on $\mathcal{R}_2 := \mathcal{P}_2$. Analogously as at the beginning of Section 2, we define the following sets $\mathcal{S}(\mathcal{P}_1), \mathcal{S}(\mathcal{P}_2), \mathcal{S}(\mathcal{P}), \mathcal{P}_\Omega$ and $\mathcal{S}(\mathcal{P}_\Omega)$, by replacing in the appropriate definitions P_i by $\mathcal{P}_i$, $(i = 1,2)$ and f by $\mathcal{F}$.

3.2 Definition of the combinatorial Ω-map

By analogy with the notion of a combinatorial map, if $\mathcal{R}_i, \mathcal{R}'_i$ are finite subsets of $\mathcal{K}$ for $i \in \{0,1,2\}$, and $\mathcal{R}_2 \subset \mathcal{R}_0 \cap \mathcal{R}_1$ and $\mathcal{R}'_2 \subset \mathcal{R}'_0 \cap \mathcal{R}'_1$, then

$$\mathcal{G}\colon \Omega_{\mathcal{R}_2}(\mathcal{R}_1,\mathcal{R}_0) \multimap \Omega_{\mathcal{R}'_2}(\mathcal{R}'_1,\mathcal{R}'_0)$$

is called a *combinatorial Ω-map.*

We also say that a combinatorial Ω-map $\mathcal{G}$ (as above) is a *combinatorial Ω-representation* of an Ω-map $g\colon |\Omega_{\mathcal{R}_2}(\mathcal{R}_1,\mathcal{R}_0)| \to |\Omega_{\mathcal{R}'_2}(\mathcal{R}'_1,\mathcal{R}'_0)|$ if

$$g([Q,i]) \subset \operatorname{int} |\mathcal{G}([Q,i])|.$$

As in [22] and [23], we define the map $\widetilde{\mathcal{F}_\mathcal{P}}\colon \mathcal{P}_\Omega \multimap \mathcal{S}(\mathcal{P}_\Omega)$ in the following way:

$$\widetilde{\mathcal{F}_\mathcal{P}}([Q,i]) := \begin{cases} \{[R,0] : R \in \mathcal{F}(Q)\} & \text{if } Q \in \mathcal{P}_2, \\ \{[R,1] : R \in \mathcal{F}(Q)\} & \text{if } Q \in \mathcal{P}_1 \setminus \mathcal{P}_2. \end{cases} \tag{13}$$

Property (a) in Definition 1.3 implies that $\widetilde{\mathcal{F}_\mathcal{P}}$ is well defined.

To shorten the notation, define $P_1 := |\mathcal{P}_1|$, $P_2 := |\mathcal{P}_2|$, and $\big|(\mathcal{R}_1,\mathcal{R}_2)\big| := \big(|\mathcal{R}_1|,|\mathcal{R}_2|\big)$ for $\mathcal{R}_1,\mathcal{R}_2 \subset \mathcal{K}$. Let us define $\widetilde{f_{\mathcal{P},\mathcal{F}}}\colon |\mathcal{P}_\Omega| \to |\mathcal{S}(\mathcal{P}_\Omega)|$ as

$$\widetilde{f_{\mathcal{P},\mathcal{F}}}([x,i]) := \widetilde{f_P}([x,i]),$$

for $[x,i] \in |\mathcal{P}_\Omega|$. Note that $\widetilde{f_{\mathcal{P},\mathcal{F}}}$ defined above and $\widetilde{f_P}$ defined by (7) differ only by codomain, and $S(P_\Omega) \subset \big|\mathcal{S}(\mathcal{P}_\Omega)\big|$. It is easy to notice that the following holds.

Proposition 3.1 *The map $\widetilde{\mathcal{F}_\mathcal{P}}$ defined by (13) is a combinatorial Ω-representation of $\widetilde{f_{\mathcal{P},\mathcal{F}}}$.*

3.3 The excision property for the combinatorial sets

We end this section with the following statement indicating that the two-layer equivalent $\mathcal{P}_\Omega$ of the combinatorial index pair is substantially better than $\mathcal{P}$ for the purpose of the Conley index computation.

Theorem 3.2 *The quadruple $\big(|\mathcal{P}_\Omega|, |\mathcal{S}(\mathcal{P}_\Omega)|\big)$ satisfies the excision property.*

The proof of this theorem is essentially the same as the proof of Theorem 2.5, and we will skip it. As it will be seen in Section 4, this property is crucial for the correctness of the Conley index computation based on the combinatorial Ω-sets.

4 Double-layer Ω-sets and the Conley index

In this section we use the Alexander-Spanier cohomology functor in order to prove that the cohomological Conley index computed with the use of double-layer Ω-sets and the corresponding Ω-map coincides with the Conley index of the original index pair.

4.1 The Conley index computed from the Ω-sets

We begin by considering an index pair $P = (P_1, P_2)$ for a continuous map f, and the corresponding Ω-sets and Ω-map. By the excision property for Alexander-Spanier cohomology (see [21], Theorem 6.6.5), Theorem 2.5 implies the following

Theorem 4.1 *The inclusion*

$$\widetilde{i_P} \colon P_\Omega \hookrightarrow S(P_\Omega) \tag{14}$$

induces an isomorphism in cohomology.

By analogy with (4), we define the homomorphism

$$\widetilde{I_P^*} \colon H^*(P_\Omega) \to H^*(P_\Omega) \tag{15}$$

as follows:

$$\widetilde{I_P^*} := \widetilde{f_P}^* \circ (\widetilde{i_P}^*)^{-1}. \tag{16}$$

We call the above map an Ω-*index map.*

Theorem 4.2 *Under the already established notation, the following diagram commutes:*

$$\begin{array}{ccc} H^*(P) & \xleftarrow{\;I_P^*\;} & H^*(P) \\ \uparrow{\scriptstyle h^*} & & \uparrow{\scriptstyle h^*} \\ H^*(P_\Omega) & \xleftarrow[\;\widetilde{I_P^*}\;]{} & H^*(P_\Omega) \end{array} \tag{17}$$

and h^* is an isomorphism.

Proof: Consider the following diagram

$$\begin{array}{ccccc} P & \xrightarrow{\;f_P\;} & S(P) & \xleftarrow{\;i_P\;} & P \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle S(h)} & & \downarrow{\scriptstyle h} \\ P_\Omega & \xrightarrow[\;\widetilde{f_P}\;]{} & S(P_\Omega) & \xleftarrow[\;\widetilde{i_P}\;]{} & P_\Omega \end{array} \tag{18}$$

where f_P, $\widetilde{f_P}$, h and $S(h)$ are defined by the formulas (1), (7), (9) and (10), respectively. The maps i_P and $\widetilde{i_P}$ are the inclusions defined by (2) and (14), respectively. By Lemmas 2.3 and 2.4, diagram (18) commutes. Therefore, after having applied the cohomology functor to (18) and inverted the isomorphisms i_p^* and $\widetilde{i_p}^*$, we obtain the following diagram that also commutes:

$$
\begin{array}{ccccc}
H^*(P) & \xleftarrow{f_{P*}} & H^*\big(S(P)\big) & \xleftarrow{(i_P^*)^{-1}} & H^*(P) \\
\uparrow{\scriptstyle h^*} & & \uparrow{\scriptstyle S(h)^*} & & \uparrow{\scriptstyle h^*} \\
H^*(P_\Omega) & \xleftarrow[\widetilde{f_P}^*]{} & H^*\big(S(P_\Omega)\big) & \xleftarrow[(\widetilde{i_P}^*)^{-1}]{} & H^*(P_\Omega)
\end{array}
\tag{19}
$$

By Proposition 2.2, Theorem 4.1, and the excision property for i_P, all maps in this diagram but f_P^* and $\widetilde{f_P}^*$ are isomorphisms. By formulas (4) and (16), the horizontal arrows in diagram (17) are compositions of maps corresponding to the horizontal arrows in diagram (19); therefore, diagram (17) also commutes. □

An immediate consequence of Theorem 4.2 is the following

Corollary 4.3 *The Leray reduction of $\big(H^*(P_\Omega), \widetilde{I_P^*}\big)$ is isomorphic to the Leray reduction of $\big(H^*(P), I_P^*\big)$, that is, the Conley index of P with respect to f.*

In other words, one can compute $H^*(P_\Omega)$ and $\widetilde{I_P}^*$ instead of $H^*(P)$ and I_P^*, respectively, and then continue with the usual procedure of applying the Leray reduction functor, in order to obtain the cohomological Conley index of P.

4.2 Computing the Conley index with the combinatorial Ω-sets

Let us now consider a combinatorial index pair $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2)$ for a combinatorial representation $\mathcal{F}$ of f. The key point in justifying the correctness of our approach is the following

Theorem 4.4 *The inclusion*

$$\widetilde{i_{\mathcal{P},\mathcal{F}}} \colon |\mathcal{P}_\Omega| \hookrightarrow |\mathcal{S}(\mathcal{P}_\Omega)|$$

induces an isomorphism in cohomology.

The proof of this theorem is essentially the same as the proof of Theorem 4.1, and we will skip it. The only difference is that now we can use a suitable excision theorem for compact polyhedra (see [15], Theorem 27.2).

Theorem 4.4 ensures that the following homomorphism

$$\widetilde{I_{\mathcal{P},\mathcal{F}}}^* := \widetilde{f_{\mathcal{P},\mathcal{F}}}^* \circ (\widetilde{i_{\mathcal{P},\mathcal{F}}}^*)^{-1} \colon H^*(|\mathcal{P}_\Omega|) \to H^*(|\mathcal{P}_\Omega|). \tag{20}$$

is well defined.

Theorem 4.5 *Under the already established notation, the following holds:*

(i) $H^*(|\mathcal{P}_\Omega|) = H^*(P_\Omega)$,

(ii) the homomorphisms $\widetilde{I_{\mathcal{P},\mathcal{F}}}^*$ defined by (20) and $\widetilde{I_P^*}$ defined by (15) are the same.

Proof: Property (i) is a straightforward consequence of the fact that $|\mathcal{P}_\Omega| = P_\Omega$.

To prove (ii) it is enough to notice that the following diagram commutes

$$\begin{array}{ccc}
H^*(P_\Omega) = H^*\big(|\mathcal{P}_\Omega|\big) & \xleftarrow{\widetilde{f_{\mathcal{P},\mathcal{F}}}^*} & H^*(|\mathcal{S}(\mathcal{P}_\Omega)|) \\
\Big\uparrow \widetilde{f_P}^* & & \Big\uparrow (\widetilde{i_{\mathcal{P},\mathcal{F}}}^*)^{-1} \\
H^*\big(S(P_\Omega)\big) & \xleftarrow[(\widetilde{i_{\mathcal{P}}}^*)^{-1}]{} & H^*(P_\Omega) = H^*\big(|\mathcal{P}_\Omega|\big)
\end{array}$$

□

The following corollary follows from Theorem 4.2 and Theorem 4.5.

Corollary 4.6 *The Leray reduction of* $\big(H^*(|\mathcal{P}_\Omega|), \widetilde{I_{\mathcal{P},\mathcal{F}}}^*\big)$ *is isomorphic to the Leray reduction of* $\big(H^*(P), I_P^*\big)$*, that is, the Conley index of* P *with respect to* f*.*

In other words, one can use $\mathcal{P}_\Omega$ and $\widetilde{\mathcal{F}_\mathcal{P}}$ to compute the cohomological Conley index of P with respect to f, without the necessity of additional verification whether the suitable inclusion map induces an isomorphism in cohomology.

5 Algorithmic computations

The main purpose of introducing the new approach to treating a combinatorial index pair and map in Section 3 was to make it possible to compute algorithmically the homological Conley index in an efficient way using full cubical sets and index pairs as in Definition 1.3. This aim has been achieved, and by Corollary 4.6 one can use our construction even if the combinatorial sets do not satisfy the excision property, as pointed out in Section 1.

In this section we explain how to use our new approach to the computation of the index map using the double-layer combinatorial sets, we illustrate the cost of using this approach in terms of computation time, and we show the advantage of this approach in comparison with using the index pairs introduced in [11]. As argued in the Introduction, we use the homology computation instead of cohomology without loss of generality, because we deal with cubical sets which are compact polyhedra.

The data for all the examples used in this paper, as well as links to related software can be found at [19].

5.1 Implementing the double-layer topology

Although in our approach one has to deal with double-layer combinatorial sets, in the actual machine computations one can essentially use the algorithms introduced in [10] for the homology computation of $|\mathcal{P}_\Omega|$ and $|\mathcal{S}(\mathcal{P}_\Omega)|$, as well as $\widetilde{\mathcal{F}_\mathcal{P}}$. The only technical issue is that a change must be made to the cubical grid structure, because the software must distinguish cubes at different layers, and one must also take into consideration the fact that in the space $|\mathcal{P}_\Omega|$ the neighborhood of a cube may look different than in $\mathcal{R}^n$, and thus the adjacency relation between cubes is slightly more complicated.

This solution is implemented in the new program `homcub2l` included in [20]. The program operates on (hyper)cubes which additionally store the information about their layer number. All the full cubes in the domain and codomain of the map are split between layer 1 ($\mathcal{X} \setminus \mathcal{A}$) and layer 0 (the complement of $\mathcal{X} \setminus \mathcal{A}$), and the program stores the boundary between $\mathcal{X} \setminus \mathcal{A}$ and $\mathcal{A}$ to switch between the layers while computing adjacent full cubes or boundaries of faces of cubes at layer 1. The effectiveness of this approach is illustrated with a few examples listed in Table 1.

example name	space dim	size of $\mathcal{X}$	`homcubes` (old prog)	`homcub2l` (new prog)	`liftcubes` + `homcubes`
Example from [1]	2	17,991	failure	2.19 sec.	3.17 sec.
Rev. Vanderpol	2	64,182	26 sec.	34 sec.	136 sec.
Ex. 1 from [5]	5	6,242	21 sec.	27 sec.	104 sec.
Ex. 2 from [5]	5	29,670	failure	98 sec.	364 sec.
Ex. 3 from [5]	6	10,330	2,220 sec.	4,412 sec.	16,309 sec.

Table 1: Computation times of the index map for a few sample index pairs. See [19] for the actual data files. The occasional failure in the computations with `homcubes` is caused by the lack of the excision property. All the computations were run on the Intel® Xeon® 5030 2.66 GHz processor.

At this point we would like to make a remark that there also exists an easy way of computing in the double-layer topology without modifying the algorithms for the homology computation of full cubical sets in $\mathbb{R}^n$ and combinatorial maps. This can be achived by embedding $\mathcal{S}(\mathcal{P}_\Omega)$ into full cubical sets in $\mathbb{R}^{n+1}$ in such a way that each cube Q at layer 0 is replaced by $Q \times [0,1]$, each cube Q at layer 1 is replaced by $Q \times [2,3]$, and each cube Q contained in the set $\mathcal{P}_2$ on which the two layers are identified is replaced by $Q \times [1,2]$. This operation can be done by the Perl script `liftcubes.pl`, and then the homology computation can be carried out by the old program `homcubes` which follows the algorithms introduced in [10], both programs available in [20]. Obviously, this approach gives rise to a slow-down in the computations, which is due to the increase in the dimension of cubes, as one can see in Table 1. More details on this alternative approach are given in a note posted at [19].

5.2 An application to real data

As an example of an application of our technique to some real data obtained in a computer-assisted proof in dynamical systems, we would like to discuss Example 2 from [5]. The problem faced there was to compute the homological Conley index of some index pair $(\mathcal{P}_1, \mathcal{P}_2)$ such that $\mathcal{P}_1$ consisted of 29 670 (hyper)cubes in $\mathbb{R}^5$, and $\mathcal{P}_2$ had 11 403 cubes. Since $H_*(|\mathcal{P}_1|, |\mathcal{P}_2|) \simeq (\mathbb{Z}, \mathbb{Z}^{21})$ and $H_*\big(\mathcal{S}(|\mathcal{P}_1|), \mathcal{S}(|\mathcal{P}_2|)\big) \simeq (\mathbb{Z}, \mathbb{Z}^{20})$, one cannot use these sets directly to compute I_{P*} with the software [20], in spite of what is claimed in [5].[3] Moreover, trying to leave in $\mathcal{P}_2$ only those cubes which are adjacent to cubes in $\mathcal{P}_1 \setminus \mathcal{P}_2$ in order to get a weak index pair as in [11] does not help here, because the excision

[3] As of writing of this paper, such data was available at the address referred to in [5]: `http://math-www.upb.de/~junge/kot_schaffer/code/ex2/`

property is still not satisfied. Therefore, our approach is necessary in this case to compute the homological Conley index using full cubical sets. Although the result of the computations differs from the one claimed in [5] (probably their computations were based on some other data), it has the expected properties which allow one to use their reasoning to arrive at the same conclusion.

5.3 Comparison of two definitions of index pairs

In the remainder of this section, we would like to discuss some examples that explain the reasons why we insist on using Definition 1.3 of a combinatorial index pair (which leads to the problems with the excision property) instead of accepting the definition suggested in [11] (which immediately implies the excision property, see Theorem 8.1 and Proposition 8.2 therein).

In a recently developed computational approach to the Conley decomposition theorem [2, 9], isolating neighborhoods of Morse sets are automatically created. In order to compute the Conley index of those sets, it is necessary to construct suitable index pairs, which may not be easy in general. However, if $\mathcal{N}$ denotes a combinatorial representation of an isolating neighborhood of a Morse set, and $\mathcal{F}$ denotes the combinatorial cubical multivalued map, then $\big(\mathcal{N}\cup\mathcal{F}(\mathcal{N}), \mathcal{F}(\mathcal{N})\setminus\mathcal{N}\big)$ satisfies Definition 1.3, which instantly solves this problem. This is especially important if the Conley index computation needs to be done for a large number of automatically generated Morse sets, like in [1]. Occasional failures caused either by the lack of the excision property of by trying to construct a more demanding index pair would have a detrimental effect on the reliability of such computations. It is worth to note that in the actual computations discussed in [1] index pairs without the excision property indeed appear several times, which proves the usefulness of our approach; one such example is mentioned in Table 1, another is illustrated in Figure 3.

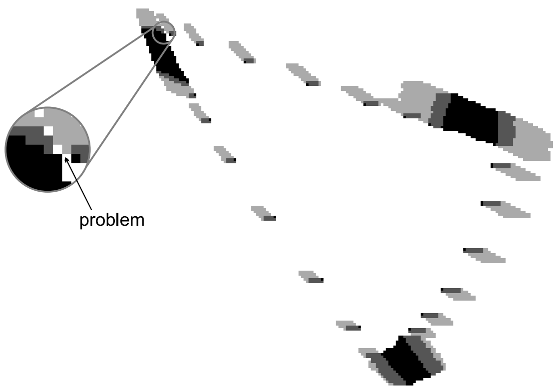


Figure 3: A sample index pair obtained in the actual computations described in [1] for a nonlinear 2-dimensional Leslie population model: $|\mathcal{P}_1\setminus\mathcal{P}_2|$ is indicated in black, $|\mathcal{P}_2|$ is shaded in dark grey, and $|\mathcal{F}(\mathcal{P}_2)\setminus\mathcal{P}_2|$ is plotted in light grey. An additional homology generator appears in $H_*\big(|\mathcal{P}_1\cup\mathcal{F}(\mathcal{P}_2)|, |\mathcal{P}_2\cup\mathcal{F}(\mathcal{P}_2)|\big)$ because of the intersection of two squares which have a common vertex indicated by an arrow in the magnified area.

In the following examples we construct a few index pairs that satisfy either our definition, or the one introduced in [11], we point out the differences, and we explain the reasons for these differences. We use the algorithm introduced in [17] to construct an index pair as in Definition 1.3. Although in [11] the author does not mention any algorithm for the construction of his index pairs, it seems to be relatively straightforward to come up with one (see [25] for some suggestions).

We consider three different maps. The first one is the well-known Hénon map

$$h : \mathbb{R}^2 \ni (x, y) \mapsto (1 + y/5 - ax^2, 5bx) \in \mathbb{R}^2$$

with $a = 1.4$ and $b = 0.2$, as in the Examples section of [11]. The other two maps are the translations by the time $\frac{71}{128}$ and $-\frac{35}{128}$, respectively, in the dynamical system induced by the Vanderpol differential equations in $\mathbb{R}^2$

$$\begin{aligned} x' &= -y + x^3 - x, \\ y' &= x. \end{aligned}$$

To make these two maps more interesting, we embed this system in $\mathbb{R}^3$ by adding an equation similar in spirit to

$$z' = -z$$

to make the plane $z = 0$ stable, so that the dynamics is essentially limited to this plane, and all the discrete trajectories in the space approach it.

We use the grid size $\frac{1}{64}$ for the Hénon map, and $\frac{1}{32}$ for the two maps that come from the Vanderpol equations. We obtain some rough approximations of the isolated invariant sets we are interested in from numerical simulations, and then we run both algorithms to construct index pairs. We compare the size of the constructed set $\mathcal{P}_1 \setminus \mathcal{P}_2$, as well as the number of cubes on which the map $\mathcal{F}$ was computed. The former is crucial for the effectiveness of the homology computation, and the latter may be very important if the map $\mathcal{F}$ comes from some expensive rigorous numerical computations. Sample results of computations are listed in Table 2.

	Hénon map		Vanderpol map		Reversed Vanderpol	
	(a)	(b)	(a)	(b)	(a)	(b)
card$(\mathcal{P}_1 \setminus \mathcal{P}_2)$	307	295	1,304	1,270	2,056	2,056
card dom $\mathcal{F}$	610	425	5,802	1,270	8,082	3,122

Table 2: Sizes of index pairs constructed with different algorithms: (a) satisfying the definition in [11], (b) satisfying Definition 1.3. The number of cubes on which the map $\mathcal{F}$ is computed is also specified.

The first noticeable advantage of our combinatorial index pair is that the map is computed on much fewer cubes. This is due to the fact that in our definition we do not impose any conditions on a cubical neighborhood of the set $\mathcal{P}_1 \setminus \mathcal{P}_2$.

The second advantage, the size of $\mathcal{P}_1 \setminus \mathcal{P}_2$, is not that profound, but in some cases may be important. The reason for this difference comes from the fact that, roughly speaking, in our index pair we do not require the isolation at the level of cubes, but rather this isolation is included in the "int" part of condition (5).

A simple program which does the computations described in this subsection is available at [19].

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