Computer Assisted Proof of the Existence of a Periodic Orbit in the Rössler Equations

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Abstract. We prove the existence of a periodic solution to a well known system of differential equations in $\mathbb{R}^3$. In the proof a topological approach using Conley index theory both for flows and for discrete dynamical systems is combined with a computer assisted construction of a guaranteed enclosure of a time-one map and computation of its homology.

In this note the following system of differential equations in $\mathbb{R}^3$, introduced in 1976 by Rössler, is discussed [6]:

$$\begin{aligned} \dot{x} &= -(y+z), \\ \dot{y} &= x + by, \\ \dot{z} &= b + z(x-a). \end{aligned} \tag{1}$$

For the parameters originally considered by Rössler, $a = 5.7$ and $b = 0.2$, the existence of chaotic dynamics has been recently proved by Zgliczyński [7].

We will focus on $a = 2.2$ and $b = 0.2$, where numerical simulations suggest existence of a periodic orbit, but—as far as we know—this fact has not been proved so far. Our aim is to fill this gap, i.e. we prove the following theorem:

Theorem 1 *For the dynamical system generated by the Rössler equations (1) with $a = 2.2$ and $b = 0.2$ there exists a periodic orbit close to the one observed in numerical simulations.*

The closeness of the orbits is understood in such a way that the periodic orbit is proved to be in an effectively constructed neighbourhood of the numerically observed one.

This note contains only a sketch of the proof, as the whole proof has a lot of technical details. We plan to publish a detailed description of the proof soon.

We start with recalling a few basic definitions and some theorems on which the proof is based.

Let $X = \mathbb{R}^3$. Write the equations (1) as $\dot{x} = f(x)$ for $x \in X$. The flow generated by these equations is a continuous dynamical system $\pi : X \times \mathbb{R} \ni (x,t) \to \pi(x,t) \in X$ such that $\pi(x,\cdot) : \mathbb{R} \to X$ is a solution to (1) with $\pi(x,0) = x$ for every $x \in X$. For $t > 0$ a discretization π_t, called also a time-t map, of the flow π is a discrete dynamical system being a restriction of π to $X \times t\mathbb{Z}$.

For a set $N \subset X$ its invariant part with respect to the flow π is defined as $\operatorname{inv} N = \{x \in N \mid \pi(x,T) \subset N\}$ with $T = \mathbb{R}$, and with respect to its t-discretization—with $T = t\mathbb{Z}$.

The set N is called an isolating neighbourhood if it is compact and $\operatorname{inv} N \subset \operatorname{int} N$. In such a case, $S = \operatorname{inv} N$ is called an isolated invariant set.

We say that an isolating neighbourhood N admits a Poincaré section if there exists a set $\Xi \subset X$ which is a local section for π, $\Xi \cap N$ is closed, and for every $x \in N$ there exists $t > 0$ such that $\pi(x,t) \in \Xi$.

The definition of the Conley index of an isolating neighbourhood N, both in the discrete and in the continuous case, is based on the notion of an index pair, i.e. an appropriate pair (P_1, P_2) of compact subsets of N. For the definition of the Conley index the reader is referred to [4] and [5]. We shall only mention that the index does not depend on the choice of an isolating neighbourhood of an isolated invariant set S. Because of this the Conley index of S may be defined as the one of its isolating neighbourhood.

The existence of a periodic orbit in our proof is a consequence the following theorem:

Theorem 2 (Corollary 1.4 from [4]) *Assume N is an isolating neighbourhood for the semiflow π defined on a metric space X which admits a Poincaré section. If N has the Conley index of a hyperbolic periodic orbit, then* $\operatorname{inv} N$ *contains a periodic orbit.*

Since we work with a discretization of the flow π, we need to use the following two theorems:

Theorem 3 (Theorem 1 from [5]) *Assume X is a metric space, $\pi : X \times \mathbb{R}^+ \to X$ is a semiflow and $S \subset X$ is compact. Then the following three conditions are equivalent:*

(1) S is an isolated invariant set with respect to π,

(2) S is an isolated invariant set with respect to π_t for all $t > 0$,

(3) S is an isolated invariant set with respect to π_t for any $t > 0$.

As a corollary from this theorem we prove that a neighbourhood N isolating for S with respect to π_t is also isolating for the same set S with respect to the flow π.

Theorem 4 (Corollary from [5]) *The cohomological Conley index of an isolated invariant set of a semiflow coincides with the corresponding index with respect to the discrete dynamical system π_t for any $t > 0$.*

In the computer assisted proof of the existence of a periodic orbit we proceeded as follows. In the beginning, computer was used to construct a set N built of cubes of fixed grid, taken along the hypothetic periodic orbit and around it where necessary to have $\pi_t(N) \subset \operatorname{int} N$ for a certain $t > 0$. This property was needed for the set N to be an isolating neighbourhood with respect to π_t. In this construction we used Lohner's algorithm for computation of guaranteed enclosures for the solutions of ordinary initial value problems [3], recently implemented by Mrozek and Zgliczyński.

Afterwards, the Conley index of N with respect to π_t was computed, with the index pair chosen to be $(N, \emptyset)$. First, a multivalued cubical map being a guaranteed enclosure of π_t on N was constructed. Then, an algorithm of Allili and Kaczyński [1] recently implemented by Mazur and Szybowski was used to transform this cubical map into a chain map which would induce the same map in homology. Finally, the homology of this chain map was computed with my own program based on an algorithm of Kaczyński, Mrozek and Ślusarek [2]. The result was that the homology of N was isomorphic to the homology of a circle, and the chain map induced the identity in homology. In such a way the Conley index of N was proved to be the one of an attracting periodic orbit.

Let us now focus on the assumptions of Theorem 2. From Theorem 4 it follows that the Conley index of N with respect to the flow π coincides with the just computed Conley index with respect to the discretization π_t of the flow π, which is obviously the one of a hyperbolic periodic orbit. It remains to check that N admits a Poincaré section. For this purpose a rectangle Ξ in the plane $y = 0$ was chosen and it was first proved manually that every positive semitrajectory starting in a certain set Q crossed Ξ, and then it was checked with computer that every positive semitrajectory starting in N fell into Q.

In this way the assumptions of Theorem 2 have been verified, and its thesis says that N contains a periodic orbit indeed, what means the end of the proof.

The method applied here may use other algorithms being recently implemented to compute discrete Conley Index of an isolating neighbourhood and to use the same theoretical basis to prove the existence of a periodic orbit even if it is not necessarily attracting. Moreover, this method is dimension-independent and thus may be used to prove the existence of a periodic orbit in $\mathbb{R}^n$ also for $n \neq 3$.

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References

[1] M. Allili, T. Kaczyński. An algorithmic approach to the construction of homomorphisms induced by maps in homology. Preprint no 205, Departement de Mathématiques et d'Informatique Université de Sherbrooke, Sherbrooke 1998.

[2] T. Kaczyński, M. Mrozek, M. Ślusarek. Homology Computation by Reduction of Chain Complexes. *Computers and Mathematics*, accepted.

[3] R.J. Lohner. Computation of Guaranteed Enclosures for the Solutions of Ordinary Initial and Boundary Value Problems, in: *Computational Ordinary Differential Equations*, J.R. Cash, I. Gladwell Eds., Clarendon Press, Oxford, 1992.

[4] C. McCord, K. Mischaikow, M. Mrozek. Zeta Functions, Periodic Trajectories and the Conley Index. *Journal of Diff. Equations*, **121**(2):258-292, (1995).

[5] M. Mrozek. The Conley Index on Compact ANR's is of Finite Type. *Results in Mathematics*, **18**:306-313, (1990).

[6] O.E. Rössler. An Equation for Continuous Chaos. *Phys. Lett.*, **57A**:397-398, (1976).

[7] P. Zgliczyński. Computer Assisted Proof of Chaos in the Rössler Equations and in the Hénon Map. *Nonlinearity*, **10**:243-252, (1997).